1. Solve $y'' - 2xy' + 8y = 0$ with $y(0) = \alpha$, $y'(0) = 0$. [See Problem 5.2.21 in Boyce and DiPrima]
Let $y = \sum_{n=0}^{\infty} a_n x^n$ then $a_0 = \alpha$ and $a_1 = 0$.
Subsituting in the equation gives $(n+2)(n+1)a_{n+2} = (2n-8)a_n$, so all odd terms vanish and the even terms vanish starting with a_6. The solution is

$$y(x) = \alpha(1 - 4x^2 + \frac{4}{3}x^4).$$

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2. Find the Laplace transform of the function $f(t) = \begin{cases} 0 & t < a \\ t-a \ , & a \le t < b \\ 0 & b \le t \end{cases}$

[See problem 6.3.9]

$$f(t) = u_a(t) * (t-a) - u_b(t) * (t-a) = u_a(t) * (t-a) - u_b(t) * (t-b) + u_b(t) * (a-b)$$

where $u_a(t)$ is the unit step function (0 for $t < a$, 1 for $t > a$).
Then using formula 13, 12 and 3 in table 6.2.1:

$$F(s) = \frac{e^{-as}}{s^2} - \frac{e^{-bs}}{s^2} + \frac{e^{-bs}}{s}(a-b).$$

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3. Solve $y'' + 2y' + 2y = u_\pi(t)$ with $y(0) = 0$, $y'(0) = \beta$. [See 6.4.2]
By Laplace transform:

$$(s^2 + 2s + 2)Y(s) - \beta = \frac{e^{-\pi s}}{s},$$

or

$$Y(s) = \frac{\beta}{s^2 + 2s + 2} + \frac{e^{-\pi s}}{s(s^2 + 2s + 2)}.$$

Now $s^2 + 2s + 2 = (s+1)^2 + 1$ so by formula 9 in table 6.2.1 the inverse transform of the β term is $\beta e^{-t} \sin t$. For the 2nd term we need to use a partial fraction expansion. Leave the exponential factor aside for now, we'll take care of it later using formula 13 in the table,

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + c}{s^2 + 2s + 2}$$

which implies

$$1 = As^2 + 2As + 2A + Bs^2 + Cs$$

or $A = 1/2$, $B = -1/2$ and $C = -1$. Then

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{1}{2s} - \frac{1}{2}\frac{s+2}{s^2 + 2s + 2} = \frac{1}{2s} - \frac{1}{2}\frac{s+1}{(s+1)^2 + 1} - \frac{1}{2}\frac{1}{(s+1)^2 + 1}$$

whose inverse Laplace transform follows directly from formula 1, 10 and 9 in the table

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 2s + 2)}\right] = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t.$$

Putting everything together, and taking care of the $e^{-\pi s}$,

$$y(t) = \beta e^{-t} \sin t + u_\pi(t) \left[\frac{1}{2} - \frac{1}{2} e^{-(t-\pi)} \left(\cos(t-\pi) + \sin(t-\pi) \right) \right],$$

$$= \beta e^{-t} \sin t + u_\pi(t) \left[\frac{1}{2} + \frac{1}{2} e^{(\pi-t)} \left(\cos t + \sin t \right) \right],$$

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4. Find the general solution and sketch the phase portrait (i.e. a representative sample of trajectories) for

$$\mathbf{x}' = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} \mathbf{x}.$$

[See 7.5.8]

$$\mathbf{x}(t) = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t.$$

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5. Solve the initial value problem and sketch the solution in the phase plane $x_1 x_2$:

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

[See 7.7.1]

$$\mathbf{x}(t) = \begin{pmatrix} 4t+4 \\ 2t+1 \end{pmatrix} e^t.$$