Consider the differential equation $\frac{d^2 Q}{dt^2} + \gamma \frac{dQ}{dt} + Q = \sin \omega t$ where $\gamma$ and $\omega$ are constants with $0 < \gamma < 1$.

(a) Find a particular solution: Constant coeffs so we try $Q_p(t) \propto \sin \omega t$ but the $dQ/dt$ brings in a $\cos \omega t$ also, so try $Q_p(t) = A \cos \omega t + B \sin \omega t$, substituting in the $Q(t)$ diff eq we get

$$(-\omega^2 A + \gamma \omega B + A) \cos \omega t + (-\omega^2 B - \gamma \omega A + B) \sin \omega t = \sin \omega t$$

now $\sin$ ans $\cos$ are linearly independent functions so the equation implies that we need

$$\begin{align*}
(1 - \omega^2) A + \gamma \omega B &= 0 \\
(1 - \omega^2) B - \gamma \omega A &= 1 
\end{align*}$$

The letters $\gamma$ and $\omega$ stand for parameters of the problem, i.e. although their numerical value is not specified they’re assumed to be known and we want to know the solution in terms of those parameters. $A$ and $B$ are undetermined constants that we have introduced to solve the problem, those are the guys we want to solve for in terms of $\gamma$ and $\omega$. That’s a little bit of algebra: the first eqn says $A = \gamma \omega B/(\omega^2 - 1)$ then the 2nd eqn gives $[(1 - \omega^2)^2 + \gamma^2 \omega^2] B = (1 - \omega^2)$ or

$$A = \frac{-\gamma \omega}{(1 - \omega^2)^2 + \gamma^2 \omega^2}, \quad B = \frac{(1 - \omega^2)}{(1 - \omega^2)^2 + \gamma^2 \omega^2}.$$  

(b) Find the general solution: The equation is linear in $Q$ so we can use superposition: $Q(t) = CQ_1(t) + DQ_2(t) + Q_p(t)$ where $Q_p(t)$ is the particular solution found in (a) and $Q_1$, $Q_2$ are two linearly independent solutions of the homogeneous eqn $\frac{d^2 Q}{dt^2} + \gamma \frac{dQ}{dt} + Q = 0$. This eqn has constant coefficients hence we look for $Q(t) = e^{rt}$ for some constant $r$. This yields the algebraic eqn $r^2 + \gamma r + 1 = 0$ whose solutions are

$$r_1 = \frac{-\gamma + i\sqrt{4 - \gamma^2}}{2}, \quad r_2 = \frac{-\gamma - i\sqrt{4 - \gamma^2}}{2}$$

where I have used $0 < \gamma < 1$ and $i^2 = -1$. So the general solution is

$$Q(t) = Ce^{-\gamma t/2} \cos \omega_0 t + De^{-\gamma t/2} \sin \omega_0 t + Q_p(t)$$

where $Q_p(t)$ is as in (a), $C$ and $D$ are constants to be determined from the initial conditions and $\omega_0 = \sqrt{1 - \gamma^2}/4$.

(c) What is the form of the solution after a long time? well $\gamma > 0$ so $e^{-\gamma t/2} \rightarrow 0$ as $t \rightarrow +\infty$, therefore $Q(t) \rightarrow Q_p(t)$. The systems “forgets” its initial conditions and the solution will eventually look mostly like the particular solution (forced response).

(d) What does “long time” mean for this equation in practice? $t \rightarrow \infty$ is a pretty long time. The Sun will have shut off long before that and no one will be around to see what happens to $Q(t)$. However we only need to wait until $\exp(-\gamma t/2)$ is “small enough” and that happens pretty quickly on a time scale of $O(1/\gamma)$. For instance $\exp(-10) \approx 4.54 \times 10^{-5}$ and that happens at $t = 20/\gamma$. If you’re willing to wait twice as long then $\exp(-20) \approx 2.06 \times 10^{-9}$. The key result is that the time is inversely proportional to $\gamma$. 

FW Math 320 Exam 2 solutions, 11/27/2002
2. [20 pts] Find the general solution to \( \frac{d^3y}{dt^3} - y = t \) then set-up, but do not solve, the system of algebraic equations that would yield the solution with \( y(0) = 1 \) and \( y'(0) = y''(0) = 0 \).

The equation is linear in \( y(t) \) so the general solution is the sum of the general homogeneous solution and a particular solution. The latter is obviously \( y_p(t) = -t \). The equation has constant coeffs so try \( y(t) = e^{rt} \) as before. That yields \( r^3 = 1 \) not to be confused with \( (r - 1)^3 = 0 \). The latter would give a repeated triple root but the former as three distinct roots, one real \( r_1 = 1 \) and a complex conjugate pair \( r_2 = r_3^* \).

To find all those roots let’s emphasize that we’re really in complex space by writing \( r^3 = 1 \) as \( r^3 = e^{2ik\pi} \) where \( k = 0, \pm 1, \pm 2, \ldots \) is any integer. Writing the complex number \( r \) in its polar representation \( r = |r| \exp(i\theta) \) so that \( r^3 = |r|^3 \exp(3i\theta) = e^{2ik\pi} \), we need \( |r| = 1 \) and \( 3\theta = 2k\pi \). This yields

\[
r = e^{2ik\pi/3}, \quad k = 0, \pm 1, \pm 2, \ldots
\]

which corresponds to three distinct roots \( r_1 = 1, r_2 = \exp(2i\pi/3) \) and \( r_3 = \exp(-2i\pi/3) \). All other values of \( k \) yield copies of these roots. You can get around this by factoring \( r^3 - 1 = (r - 1)(r^2 + r + 1) \) but would do well to understand the ‘complex’ approach.

(Exercises: find the roots of \( r^{17} = 1 \); find the roots of \( r^9 = 8 \).

So the general solution is

\[
y(t) = Ae^{r_1t} + Be^{r_2t} + Ce^{r_3t} - t.
\]

Imposing the initial conditions leads to the algebraic system

\[
\begin{align*}
A + B + C &= 1 \\
r_1A + r_2B + r_3C &= 1 \\
r_1^2A + r_2^2B + r_3^2C &= 0
\end{align*}
\]

This system has a unique solution because the determinant

\[
\begin{vmatrix}
1 & 1 & 1 \\
r_1 & r_2 & r_3 \\
r_1^2 & r_2^2 & r_3^2
\end{vmatrix}
= (r_2 - r_1)(r_3 - r_1)
\begin{vmatrix}
1 & 0 & 0 \\
r_1 & 1 & 1 \\
r_1^2 & r_2 - r_1 & r_3 - r_1
\end{vmatrix}
= (r_2 - r_1)(r_3 - r_1)(r_3 - r_2) \neq 0
\]

as the roots are distinct.

3. [10 pts] The eigenvalues and eigenvectors of the matrix

\[
A = \begin{bmatrix}
0 & 4 \\
-1 & 0
\end{bmatrix}
\]

are \( \lambda = 2i \) with \( v = [2, i]^T \) and their complex conjugate.