Detailed discussion of **'charged particle moving in a constant magnetic field,'** or in mathematical terms

$$\frac{d^2\vec{\boldsymbol{r}}}{dt^2} = \frac{d\vec{\boldsymbol{v}}}{dt} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}} \tag{1}$$

where $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}(t) = d\vec{\boldsymbol{r}}/dt$.

Physics remark: we have absorbed all the scalar constants like electric charge q and mass m into the $\vec{\boldsymbol{b}}$. You can think of $\vec{\boldsymbol{b}}$ as the 'magnetic field' but it's actually the magnetic field times some constants that have their own physical units. If you know about the Lorentz force from E&M then $\vec{\boldsymbol{b}} = (-q/m)\vec{\boldsymbol{B}}$. What are the physical units of $\vec{\boldsymbol{b}}$ in equation (1)? [Hint: the equation tells you!]

Goal: Equation (1) is the vector differential equation for $\vec{\boldsymbol{v}}(t)$. We want to solve it for $\vec{\boldsymbol{v}}(t)$. The constant vector $\vec{\boldsymbol{b}}$ is 'known'.

The first thing we notice is that $d\vec{\boldsymbol{v}}/dt = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}$ is always perpendicular to both $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{v}}$.

▷ Perpendicularity of $\vec{\boldsymbol{v}}$ and $d\vec{\boldsymbol{v}}/dt$ means that

$$2\vec{\boldsymbol{v}} \cdot \frac{d\vec{\boldsymbol{v}}}{dt} = 0 = \frac{d|\vec{\boldsymbol{v}}|^2}{dt} \tag{2}$$

(because $|\vec{\boldsymbol{v}}|^2 = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}$ and $d(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}})/dt = \dot{\vec{v}} \cdot \vec{\boldsymbol{v}} + \vec{\boldsymbol{v}} \cdot \dot{\vec{v}} = 2\vec{\boldsymbol{v}} \cdot \dot{\vec{v}}$). So the *magnitude* of $\vec{\boldsymbol{v}}$ does not change but its *direction* changes.

▷ Next, 'dot' the equation (1) with $\hat{\boldsymbol{b}}$ since $\vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}$ is also perpendicular to $\vec{\boldsymbol{b}}$, and

$$\hat{\boldsymbol{b}} \cdot \frac{d\vec{\boldsymbol{v}}}{dt} = \hat{\boldsymbol{b}} \cdot \left(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}\right) = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\hat{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}}\right) = 0 \tag{3}$$

(we can take the $\hat{\boldsymbol{b}}$ inside the d/dt because the $\vec{\boldsymbol{b}}$ direction is constant since $\vec{\boldsymbol{b}}$ is constant). This is telling us that the component of $\vec{\boldsymbol{v}}$ in the direction of $\vec{\boldsymbol{b}}$ does not change.

This is a big step because it says that all the 'action' is happening in the plane perpendicular to $\vec{\boldsymbol{b}}$. So write $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_{\parallel} + \vec{\boldsymbol{v}}_{\perp}$, where $\vec{\boldsymbol{v}}_{\parallel} = \hat{\boldsymbol{b}}\,(\hat{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}})$ is constant as shown in (3), then eqn (1) implies

$$\frac{d\vec{\boldsymbol{v}}_{\perp}}{dt} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_{\perp}. \tag{4}$$

You can picture this more easily. Take $\vec{\boldsymbol{b}}$ perpendicular to your sheet of paper, then $\vec{\boldsymbol{v}}_{\perp}$ and $\vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_{\perp} = d\vec{\boldsymbol{v}}_{\perp}/dt$ are all in the plane of the paper (think about it! draw it!). As before this equation means that $|\vec{\boldsymbol{v}}_{\perp}|$ is a constant (why was that again?). So we know almost everything about $\vec{\boldsymbol{v}}(t)$, its component parallel to $\vec{\boldsymbol{b}}$ does not change and its component perpendicular to $\vec{\boldsymbol{b}}$ has constant magnitude. Constant magnitude makes us think 'circle'. What 'circle' though?

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(1) *Smart component approach:* To solve (4) take *any* fixed unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ such that $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{b}}$ form a right-handed orthonormal basis, then let $\vec{\boldsymbol{v}}_{\perp} = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}}$ (why can we write that?). Since $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are fixed, (4) becomes

$$\frac{d\vec{\boldsymbol{v}}_{\perp}}{dt} = \dot{x}\,\hat{\boldsymbol{x}} + \dot{y}\,\hat{\boldsymbol{y}} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_{\perp} = |\vec{\boldsymbol{b}}|\,(x\,\hat{\boldsymbol{y}} - y\,\hat{\boldsymbol{x}}) \tag{5}$$

(why?) which splits into the two coupled ODEs (why?)

$$\dot{x} = -\omega\, y, \qquad \dot{y} = \omega\, x \tag{6}$$

where $\omega = |\vec{\boldsymbol{b}}|$ is a (positive) constant. You should make a mental note of that system, it's a fundamental one. You can eliminate one of the unknown functions, $y(t)$ say, to get $\ddot{x} + \omega^2 x = 0$. This is the *harmonic oscillator* equation. Its two linearly independent solutions are $x(t) = \cos\omega t$ and $x(t) = \sin\omega t$ and the general solution is $x(t) = A\cos\omega t + B\sin\omega t$ for some constants A and B to determine from the initial conditions. We can also write the general solution in the form $x(t) = C\cos(\omega(t - t_0))$ and take C and t_0 as the 'constants of integration'. But if we write it in that form then $x_0 = x(t_0) = C$ and $\dot{x}(t_0) = 0$ which implies $y(t_0) = 0$ from (6). So we can write the solution in the form $x(t) = x_0 \cos\omega(t - t_0)$ and from the first equation in (6), $y(t) = x_0 \sin\omega(t - t_0)$, for some constants x_0 and t_0. But this choice of constants of integration means that $\vec{\boldsymbol{v}}_\perp(t_0) = x_0\hat{\boldsymbol{x}}$ since by definition $\vec{\boldsymbol{v}}_\perp(t) = x(t)\hat{\boldsymbol{x}} + y(t)\hat{\boldsymbol{y}}$. So why bother with those $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ at all? we can write

$$\boxed{\vec{\boldsymbol{v}}_\perp(t) = \cos\varphi\; \vec{\boldsymbol{v}}_\perp(t_0) + \sin\varphi\; \left(\hat{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_\perp(t_0)\right)} \tag{7}$$

where $\varphi = |\vec{\boldsymbol{b}}|\,(t - t_0)$. (Substitute back into (4) to check and digest the solution (7)).

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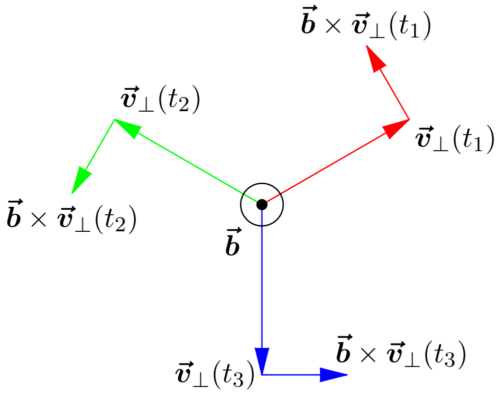
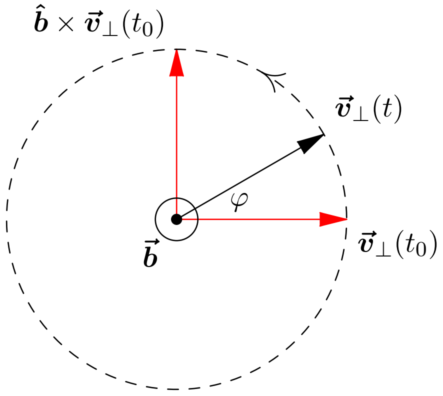
(2) *Direct vector approach:* We can actually deduce (7) 'directly' from eqn (4) since the latter implies that $|\vec{\boldsymbol{v}}_\perp(t)| = |\vec{\boldsymbol{v}}_\perp(t_0)|$ (*i.e.* the *magnitude* is constant). Furthermore the magnitude of $\vec{\boldsymbol{b}}$ is constant and the angle between $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{v}}_\perp$ is constant (it's $\pi/2$). So $|\vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_\perp| = |\vec{\boldsymbol{b}}|\,|\vec{\boldsymbol{v}}_\perp| = \mathit{constant}$ and the direction of $\vec{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_\perp$ is always $\pi/2$ counterclockwise around $\vec{\boldsymbol{b}}$ from that of $\vec{\boldsymbol{v}}_\perp$ as shown in the following figure on the left.





But this clearly means that $\vec{\boldsymbol{v}}_\perp$ is rotating counterclockwise about $\vec{\boldsymbol{b}}$ at angular velocity $|\vec{\boldsymbol{b}}|$. So if we put the tail of the velocity vector $\vec{\boldsymbol{v}}_\perp(t)$ at the same point for all t, then its head moves along a circle of radius $|\vec{\boldsymbol{v}}_\perp(t_0)|$ at angular velocity $|\vec{\boldsymbol{b}}|$. We can use $\vec{\boldsymbol{v}}_\perp(t_0)$ as one basis vector and $\hat{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_\perp(t_0)$ as the other orthogonal vector, and write the solution (7) directly with $\varphi = |\vec{\boldsymbol{b}}|\,(t - t_0)$. This should be obvious from the figure if you remember your basic trigonometry.

Putting it all together: remember that $\vec{\boldsymbol{v}}_{\|}(t) = \vec{\boldsymbol{v}}_{\|}(t_0)$ so we could actually write the full solution to (1) in vector form as

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{v}}_{\|}(t_0) + \cos\varphi\, \vec{\boldsymbol{v}}_{\perp}(t_0) + \sin\varphi\, \left(\hat{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_{\perp}(t_0)\right) \tag{8}$$

but we can also clean this up to express it in terms of the full $\vec{\boldsymbol{v}}(t_0) = \vec{\boldsymbol{v}}_0$. How? well, first $\vec{\boldsymbol{v}}_{\|} = \hat{\boldsymbol{b}}(\hat{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}})$, then $\vec{\boldsymbol{v}}_{\perp} = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_{\|}$ and $\hat{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_{\perp} = \hat{\boldsymbol{b}} \times \vec{\boldsymbol{v}}$ (why?), so (8) can be written

$$\boxed{\vec{\boldsymbol{v}}(t) = \hat{\boldsymbol{b}}\,(\hat{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}}_0) + \cos\varphi\, \left(\vec{\boldsymbol{v}}_0 - \hat{\boldsymbol{b}}\,(\hat{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}}_0)\right) + \sin\varphi\, \left(\hat{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_0\right)} \tag{9}$$

A bit scary at first, but this is a very clean and general way to write the solution. Everything is a constant in there except $\varphi = |\vec{\boldsymbol{b}}|(t - t_0)$. Note that each term has units of velocity as should be (what are the units of φ?).

Now if we want to get the particle trajectory $\vec{\boldsymbol{r}}(t)$ we're in pretty good shape since we know $\vec{\boldsymbol{v}}(t) = d\vec{\boldsymbol{r}}/dt$ explicitly. We only need to integrate (9) with respect to t to obtain

$$\vec{\boldsymbol{r}}(t) = t\,\hat{\boldsymbol{b}}\,(\hat{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}}_0) + \frac{1}{|\vec{\boldsymbol{b}}|} \sin\varphi\, \left(\vec{\boldsymbol{v}}_0 - \hat{\boldsymbol{b}}\,(\hat{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}}_0)\right) - \frac{1}{|\vec{\boldsymbol{b}}|} \cos\varphi\, \left(\hat{\boldsymbol{b}} \times \vec{\boldsymbol{v}}_0\right) + \vec{\boldsymbol{r}}_c \tag{10}$$

where $\vec{\boldsymbol{r}}_c$ is a constant of integration which we could figure out in terms of the position at t_0 (is $\vec{\boldsymbol{r}}_c = \vec{\boldsymbol{r}}(t_0)$?). This is the equation of a *helix* whose axis is parallel to $\vec{\boldsymbol{b}}$. The first term is the uniform (constant speed) motion along the direction $\hat{\boldsymbol{b}}$, the next two terms are the rotation at rate $|\vec{\boldsymbol{b}}|$ in the directions perpendicular to $\vec{\boldsymbol{b}}$, and the constant term $\vec{\boldsymbol{r}}_c$ is a point on the axis of rotation for that particle as a function of its initial conditions. *Exercises:* Find $\vec{\boldsymbol{r}}_c$ in terms of the position $\vec{\boldsymbol{r}}_0$ and velocity $\vec{\boldsymbol{v}}_0$ at time t_0. What is the equation of the axis of the helix?

IMPORTANT REMARKS, EXTENSIONS AND ANALOGIES

All of this is quite fundamental because the underlying math is not just for the motion of a charged particle in a constant magnetic field. We get the same equation if we have *uniform rotation about an axis.* For instance if a particle rotates at constant angular velocity ω counterclockwise about an axis parallel to the direction $\vec{\boldsymbol{n}}$ (unit vector) that passes through the origin, then its velocity is

$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \;\Leftrightarrow\; \frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \tag{11}$$

where $\vec{\boldsymbol{\omega}} = \omega\vec{\boldsymbol{n}}$. This is the same equation as (1) *but with $\vec{\boldsymbol{r}}$ instead of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{\omega}}$ instead of $\vec{\boldsymbol{b}}$.* The solution of this differential equation for $\vec{\boldsymbol{r}}(t)$ is that the component of $\vec{\boldsymbol{r}}$ in the $\vec{\boldsymbol{\omega}}$ direction does not change, and the perpendicular component rotates counterclockwise about the $\vec{\boldsymbol{n}}$ axis at rate ω. So mathematically it is the same problem although in the Lorentz force case it's the velocity that is 'rotating about an axis' (and as a result the particle moves along a helix while here it is the particle which rotates about an axis passing through the origin and the particle moves along a circle.)

More generally if $\vec{\boldsymbol{u}}(t)$ is *any* vector function of t and if its equation is

$$\boxed{\frac{d\vec{\boldsymbol{u}}}{dt} = \vec{\boldsymbol{c}} \times \vec{\boldsymbol{u}}} \tag{12}$$

where $\vec{\boldsymbol{c}}$ is constant, then the solution $\vec{\boldsymbol{u}}(t)$ corresponds to the rotation of $\vec{\boldsymbol{u}}$ counterclockwise about the direction $\hat{\boldsymbol{c}}$ at angular rate $|\vec{\boldsymbol{c}}|$.

Generalizing a bit, if a particle rotates at ω about an axis parallel to $\vec{\boldsymbol{n}}$ that goes not through the origin but through a point $\vec{\boldsymbol{r}}_a$ then its velocity is

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_a) \tag{13}$$

Indeed you easily check that $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_a| = constant$ (how do you 'easily check' that?) and that you can rewrite the equation in the form

$$\frac{d}{dt}(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_a) = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_a) \tag{14}$$

since $\vec{\boldsymbol{r}}_a$ is a constant (but $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}(t)$ is not constant, and $|\vec{\boldsymbol{r}}|$ is not constant either).

This brings us back to (1) which in Newton's dot notation reads $\ddot{\vec{r}} = \vec{\boldsymbol{b}} \times \dot{\vec{r}}$ but since $\vec{\boldsymbol{b}} = constant$ then $d(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{r}})/dt = \vec{\boldsymbol{b}} \times \dot{\vec{r}}$ and we can integrate the equation once in time to obtain

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{b}} \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{c}} \tag{15}$$

where $\vec{\boldsymbol{c}}$ is a constant of integration which is in fact a velocity. This looks a lot like (13) except for that (vector) constant $\vec{\boldsymbol{c}}$. But we could write it in the form (13) by simply defining $\vec{\boldsymbol{c}} = -\vec{\boldsymbol{b}} \times \vec{\boldsymbol{r}}_a$. Can we really do that? $\vec{\boldsymbol{c}}$ is an arbitrary constant of integration that would be determined by initial conditions, since $\vec{\boldsymbol{b}}$ is known we can in principle find $\vec{\boldsymbol{r}}_a$ in terms of whatever $\vec{\boldsymbol{c}}$ is. Well, not quite. Remember the problem $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ (section 1.6.3) that Megan Sharrow solved for us. The vector $\vec{\boldsymbol{b}} \times \vec{\boldsymbol{r}}_a$ has to be perpendicular to $\vec{\boldsymbol{b}}$, but $\vec{\boldsymbol{c}}$ does not have to. So we cannot always find a $\vec{\boldsymbol{r}}_a$ such that $\vec{\boldsymbol{b}} \times \vec{\boldsymbol{r}}_a = -\vec{\boldsymbol{c}}$ for any $\vec{\boldsymbol{c}}$. But we can find a $\vec{\boldsymbol{r}}_a$ and a $\vec{\boldsymbol{c}}_{\|}$ parallel to $\vec{\boldsymbol{b}}$ such that

$$\vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}_{\|} - (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{r}}_a). \tag{16}$$

so we can always write (15), the first integral of (1), in the form

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{b}} \times (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_a) + \vec{\boldsymbol{c}}_{\|} \tag{17}$$

Now the educated reader says *'Oh! the velocity is simply the sum of a rigid body rotation about an axis parallel to* $\vec{\boldsymbol{b}}$ *passing through* $\vec{\boldsymbol{r}}_a$ *and a uniform translation at speed* $\vec{\boldsymbol{c}}_{\|}$ *in the direction of* $\vec{\boldsymbol{b}}$*.'* You want to think, digest and practice all this to get to the point where you readily 'see' what equation (17) means.

Exercise: Find $\vec{\boldsymbol{r}}_a$, $\vec{\boldsymbol{c}}_{\|}$ and the radius of the cylinder on which the particle moves (the *gyroradius*) in terms of the particle position $\vec{\boldsymbol{r}}_0$ and velocity $\vec{\boldsymbol{v}}_0$ at time t_0. Work from eqn (17) *not* from (10).

So the equations (1), (12) and (14) are key 'building blocks'. The solution (9) is also quite important for other applications.

Suppose you were given this problem: rotate vector $(4, 1, 6)$ by $\pi/3$ about the direction $(1, 2, 3)$. How would you tackle that?!

(This could be part of a computer graphic problem: you have a bunch of edges representing a structure for instance and you want to rotate all these edges about an axis, for visualization purposes).

Well, if $\vec{\boldsymbol{v}}_0$ is the vector you want to rotate and $\vec{\boldsymbol{b}}$ is the rotation direction then equation (9) gives you the general formula that does the trick for any angle φ!

$\triangleright$ So what's the rotation of $(4, 1, 6)$ by $\pi/3$ about $(1, 2, 3)$?! figure it out!