A (real) random variable \( x \) is defined by its \emph{probability density function} (or \emph{probability distribution function}) \( p(x) \geq 0 \).

The probability for \( x \) to be in the interval \([x, x + dx]\) is \( p(x)dx \). So the probability that \( x \) is in the interval \([a, b]\) is \( \int_a^b p(x)dx \). The random variable must be in \([ -\infty, \infty]\) meaning that the probability to be in that infinite interval is one: \( \int_{-\infty}^{\infty} p(x)dx = 1 \).

Those two properties \( p(x) \geq 0 \) and \( \int_{-\infty}^{\infty} p(x)dx = 1 \) are the defining properties of a pdf. A function that does not satisfy those cannot be a pdf.

\[
\text{Mean} = \bar{x} = \int_{-\infty}^{\infty} x p(x)dx
\]
\[
\text{Variance} = \int_{-\infty}^{\infty} (x - \bar{x})^2 p(x)dx
\]

Examples:

\textbf{Uniform distribution:}

\[ p(x) = 1 \text{ if } 0 < x < 1, p(x) = 0 \text{ otherwise} \]

Mean=0.5, Standard deviation (STD)= \( \sqrt{\text{variance}} = 1/(2\sqrt{3}) \). Matlab’s \textbf{rand}.

\textbf{Gaussian distribution:}

\[ p(x) = \frac{e^{-(x-x_0)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \]

Mean=\( x_0 \), variance=\( \sigma^2 \), STD=\( \sigma \). Matlab’s \textbf{randn} gives the normalized distribution with mean=0 and STD=1. To get mean \( x_0 \) and std \( \sigma \), use \( x = x_0 + \sigma \cdot \text{randn} \).

The probability to be in a certain interval \([a, b]\) can be reduced to an integral of \( \exp(-u^2) \) by the change of variables \( x = x_0 + \sqrt{2}\sigma u \). That integral is directly related to the \textit{Error function} which is a Matlab built-in special function (\textbf{erf}, see “help erf”, or “help specfun”). The error function is defined as

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}du. \]

This integral cannot be done analytically. Matlab computes it numerically. For example, the probability to be within 2\( \sigma \) from the mean is

\[ P(|x - x_0| \leq 2\sigma) = \int_{x_0 - 2\sigma}^{x_0 + 2\sigma} \frac{e^{-(x-x_0)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}dx = \frac{1}{\sqrt{\pi}} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-u^2}du = \text{erf}(\sqrt{2}) \approx 0.9545. \]

This is a noteworthy fact about Gaussian distributions: there is about 95\% chance to be within 2 STD of the mean.