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# Using a Theory of Learning in College Mathematics Courses

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Suppose you were teaching a mathematics course that included the topic of coset of a group. More specifically, you might want your students to understand what a coset is, to be able to find examples and non-examples, to establish some properties such as the lemmas leading up to Lagrange's theorem as well as the theorem itself together with its applications. You might even want your students to be able to construct examples of quotient groups and prove theorems relating properties of a quotient group to properties of the original group.

There is no great difficulty in explaining all of these ideas to your students. You can give them definitions and examples. You can state theorems and give the proofs, or ask them to discover some of the arguments. You can go over various applications of the concept of cosets. All of this is done, traditionally, through lectures and classroom discussions. Unfortunately, there is a certain amount of evidence, both experimental and anecdotal, that this does not work very well. Not only do relatively few students learn the concept of cosets very well, but this topic tends to drive them away from studying mathematics.

I would like to suggest a different approach. I believe that it is possible, through a program of research, to find out something about what might be going on in a student's mind when he or she is trying to learn a mathematics topic and design instruction focused, not directly on the mathematics, but on some model of how the topic in question can be learned. This is where theory comes in. A rich theory of learning can give direction to an often confusing endeavour. It can be a basis for generalisation. Indeed, in my opinion, the coordination of a theoretical approach, instructional treatment based on that theory, and the gathering and analysis of data makes for the highest quality and most practical research in learning.

A theoretical analysis can propose mental constructions for students to make in order to learn a particular mathematical concept; a highly non-traditional pedagogy can try to get students to make those constructions in their minds, and assessment can tell us something about the relation between the predicted mental constructions and those the students appeared to have made. Of course, in using such an approach, one should not forget the main role of assessment—to tell us about the mathematics that the students may or may not have learned.

I would like to describe in this brief essay one particular theory, APOS Theory, and how it is used in such a program. There is now a fairly large body of information about the results of this approach for which I will provide some pointers to the literature. Before describing this specific example of a theoretical approach to curriculum development, I want to make some general remarks about the nature and role of theories of learning in mathematics education. Then I will outline the main points of APOS Theory and explain one way of using it in collegiate mathematics courses. Finally, I will give some information about results.

## ***What is a theory of learning and what characteristics should it have?***

These two questions are fairly easy to answer in mathematics where we have standards of rigor and "theory" consists of definitions, examples, counter examples, theorems and proofs. The characteristics of a theory are that it

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should help solve problems, prove new theorems, and make applications, both in and out of mathematics. In mathematics education, however, the matter is not so clear and there can be controversy. Because there are so many discourses which some would like to call theories, I would prefer to avoid comparisons between theories in this essay. Rather, I will describe the characteristics that I think a learning theory should have. In subsequent sections I will describe APOS Theory which is the theory that is most important in my work and explain the extent to which it possesses these characteristics.

### **Characteristics of a Theory**

**Support prediction.** A theory should help us say that if certain phenomena, call them antecedents, are observed, then other phenomena, are likely to occur as consequences. Ideally, these phenomena should be observable. Moreover, antecedents should be of such a nature that it is possible, through appropriate instructional treatments, to foster their occurrence in students. Finally, of course, the consequent phenomena should consist, essentially, of mathematical knowledge and understanding.

**Possess explanatory power.** It should be possible to use the theory to explain, in both coarse and fine grain, specific successes and failures of individuals and groups of students in trying to learn mathematical topics.

**Be applicable to a broad range of phenomena.** It is not enough to observe a phenomenon, or even a small set of phenomena and then develop a theory to connect and observe them. It should be possible to apply a theory to phenomena very different from the ones used to develop it.

**Help organise thinking about learning phenomena.** Thinking about learning tends to be ad hoc, anecdotal, and restricted to descriptions of an individual's experiences as learner and/or teacher. Scientific investigation of a domain, such as learning mathematics, requires an organised structure, including definitions of theoretical concepts and relationships among them, that practitioners discipline themselves to follow.

**Serve as a tool for analysing data.** One way to analyse a set of data is to immerse oneself in it, discuss it within a research team and use one's best thinking to make sense of it. A theory, however, should provide a more systematic method of analysis. It should tell the researchers what questions to ask of the data and how to interpret the answers.

**Provide a language for communication about learning.** Research and curriculum development must go beyond a single person or team making investigations and obtaining results. The work must be communicated and this is best done if there is a generally accepted common language. A theory can provide such a communication tool.

### **What is APOS Theory?**

APOS Theory is a constructivist theory of how learning a mathematical concept might take place. It is based on the following hypothesis about the nature of mathematical knowledge and how it is developed:

*An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations and their solutions by reflecting on them in a social context and constructing or reconstructing mathematical actions, processes and objects and organising these in schemas to use in dealing with the situations.*

In addition to our theoretical perspective, this statement is also the basis for our instructional treatment and how we gather and analyse the data. We will focus in this section on the relation of the statement to our theory. The other parts of the statement will relate to discussions in subsequent sections.

APOS Theory is our elaboration of the mental constructions of actions, processes, objects and schemas. In studying how students might learn a particular mathematical concept, an essential ingredient which the researcher must provide is an analysis of the concept in terms of these specific constructs. The description resulting from this analysis is called a *genetic decomposition* of the concept. We will consider in later sections how this decomposition is made. Here we will give an informal description of a genetic decomposition of the concept of cosets in terms of actions, processes, objects and schemas.

According to this genetic decomposition, learning cosets would begin with an *action* conception consisting of forming cosets which can be described by listing their elements, such as the cosets of the subgroup of multiples of 4 in the group of integers mod 24 with addition mod 24 as the operation. Calculations could be made of specific cosets and properties such as the number of elements in a coset, the number of cosets of a given subgroup, and disjointness could be observed. At this level of understanding, the student would not be able to consider larger and more complicated groups and a binary operation on the set of cosets of a subgroup

would be hard to understand. The idea of a quotient group would be essentially inaccessible.

A higher level of understanding would be a *process* conception in which the individual interiorizes the actions. He or she no longer requires explicit listings and calculations to perform operations but can imagine them or run through them mentally. Thus given a group  $G$ , a subgroup  $H$ , and an element  $x \in G$ , the individual can think of running through the elements of  $H$  and forming the group product of each with  $x$  (on the left, say). Not only does this allow the individual to work with more complicated groups, such as the group of permutations of  $n$  objects, but he or she can also think about other processes such as figuring out cardinalities and forming the set of all products of two group elements, the first from one specific coset and the second from another.

In moving towards thinking about properties of cosets and performing actions on them, the individual must encapsulate the process understanding and develop an *object* conception of cosets. Then it becomes possible to consider theorems concerning the cardinality of every coset of a particular subgroup, the number of cosets and their disjointness.

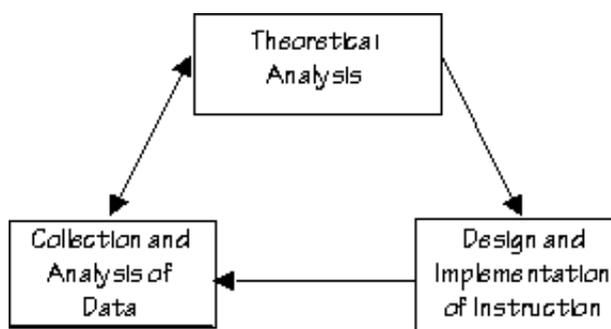
At this point, major ideas such as Lagrange's theorem become accessible to the student and he or she can think of constructing a binary operation on the set of cosets of a subgroup, determining its properties and thereby getting to the ideas of normality and quotient groups. In working with these concepts, it is important for the individual to pass easily from the object back to the process from which it came and then return to the object as needed to work with particular situations. Thus, having constructed (mentally) the set of cosets of a subgroup, the individual can think about a binary operation which takes two cosets and produces another coset. Such thoughts require an object conception of cosets. But to actually construct the binary operation (as the set of all products of two elements, one from each coset), the individual must return to the process conception. In many mathematical activities, it is necessary to go back and forth between process and object conceptions of a mathematical entity.

All of these conceptions of cosets together with properties that the individual understands are organised in what we call the individual's *schema* for cosets. A schema is a collection of actions, processes, objects and other schemas, together with their relationships, that the individual understands in connection with cosets. This collection will be coherent in the sense that the individual will have some means (explicit or implicit), perhaps the

formal definition, of determining, for any phenomenon encountered, what relationship it has to her or his conception of cosets.

### **How is APOS Theory used?**

Theoretical analysis is one component in a general program of research and curriculum development. This program functions according to a paradigm that is illustrated in Figure 1.



**Figure 1 Paradigm illustrated**

According to this paradigm, the work begins with a theoretical analysis as described in the previous section. Initially, this analysis is based on the researchers' knowledge of the concept in question and of the general theory. Our investigations cycle through the steps of this paradigm and when it is repeated, the theoretical analysis makes use of the data obtained from the previous cycle.

The purpose of the theoretical analysis is to propose specific mental constructions (the genetic decomposition) through which a student might learn the concept under consideration. The role of the instructional treatment is to get students to make the proposed mental constructions and use them to construct an understanding of the concept as well as apply it in both mathematical and non-mathematical situations. The pedagogical strategies for doing this include having students write computer programs to implement mathematical ideas, cooperative learning, and a de-emphasis of lecturing in favour of students working in groups to complete mathematical tasks. In the next section, I give a brief sketch of how this might look for the concept of cosets. The analysis of data relates to the theoretical analysis in two directions. First, the analysis provides the questions to ask of the data. On the other hand, the data tells something about the effectiveness of the theoretical analysis in terms of mental constructions. Of course the data also tells something about the

mathematics that the students may or may not have learned.

### **Instructional treatment for cosets**

Several specific pedagogical strategies are used to help the students make the desired mental constructions. The most important of these are having students write computer code to implement mathematical concepts, and cooperative learning.

As discussed in Dubinsky (1995), computer activities can foster the mental constructions called for by the theoretical analysis. Cooperative learning provides a social context in which the students can engage in the reflection referred to in the hypothesis given in the previous section. In addition, there is an attempt to have the students engage in active learning, to figure things out on their own rather than rely on the instructor for explanations and information. This results in less lecturing than in a traditional class. Inputs from the instructor are designed to further help students make the mental constructions. When the students have had an ample opportunity to make these constructions, the instructor may explain various concepts and methods to a small group of the students or to the whole class. These explanations are, again, based on the mental constructions the students are expected to have made. All of this tends to lead to an overall atmosphere that, for many students, is more supportive of learning than a traditional classroom.

The instructional treatment is organised in what is called the *ACE Teaching Cycle* of Activities to be done on the computer, Classroom discussions, and Exercises to be done with pencil and paper. A typical class will meet one or two days each in a computer lab and two or three days in a classroom.

In the computer lab, the students write computer code and programs and use these constructs to perform mathematical tasks. The computer activities are specifically designed with the mental constructions in mind. For example, to help the students construct an action understanding of cosets, they might be asked to write code such as:

```
G := {0..19};
op := |x,y → (x + y) mod 20|;
      7 .op 15;
H := {0,4,8,12,16};
K := {3,7,11,15,19};
```

where the first two lines construct the additive group of integers mod 20, the third line calculates one example

of the operation in this group, the next line forms a certain subgroup  $H$  and the last line gives the coset  $K=3 + H$ .

For a process conception, the student might write,

```
G := {[a,b,c,d] : a,b,c,d in {1..4} |
      #{a,b,c,d} = 4};
op := |p,q → [p(q(i)) : i in [1..4]]|;
H := {[1,2,3,4], [2,1,4,3], [3,4,1,2],
      [4,3,2,1]};
K := {[2,3,1,4] .op p : p in G};
      {p .op q : p in H, q in K};
```

where this time the code constructs the group of permutations of four numbers with composition as the operation, the subgroup  $H$  which is isomorphic to the Klein 4-group, a coset  $K$  of  $H$  and the coset product  $HK$ .

Finally, to help the students construct an object conception of cosets, they might be asked to write a program that will accept any two cosets and return their product. It would look like,

```
CP := func(C1, C2);
return {x .op y | x in C1, y in C2};
end;
```

where it is assumed that a group operation **op** has been defined. This program can then be applied by the student to specific examples and used to investigate properties that the coset product may or may not possess, such as commutativity.

For more information on this way of using computers, see Dubinsky (1995).

In the classroom sessions, the students are given specific mathematical tasks to perform, based on the mental constructions they have made in the computer lab. For example, they might be asked to make a general statement about the number of elements in a coset or the intersection of two cosets, based on the examples they have worked with on the computer. Then the class could move to a proof of various properties that the students have observed empirically. In addition to working on these tasks, from time to time the students will listen to explanations (or brief lectures) by the instructor, who must decide when to let the students try to figure out something on their own and when they are ready to hear an explanation. This interplay between discovery and explanation is where the teacher has the greatest opportunity to control the pace of the course and apply her or his pedagogical creativity.

Finally, exercises are assigned to do as homework. These are fairly traditional drill and practice as well as problems that require deeper thought. It is important to note that, unlike traditional instruction, the number of illustrative examples is minimised until the students have had ample opportunity to construct understandings of the mathematics involved. The reinforcement that comes from practice is an important part of learning, but it should not take place until the possibility of reinforcing misconceptions is reduced as much as possible.

All of the students' work in the course in the computer lab, in class, on the exercises and even some of the examinations, is done in cooperative groups which are established at the beginning of the course and not changed thereafter.

### **Characteristics of APOS Theory**

In this section I would like to go back to the Characteristics of a Theory described earlier and consider the extent to which APOS Theory possesses these properties.

**Support prediction.** The predictive power of APOS Theory lies in the assertion that if the student makes certain mental constructions, then he or she will learn a certain mathematical topic. This has turned out to be the case in a number of studies which are summarised in Weller *et al* (In review).

**Possess explanatory power.** Our method of analysing data, as described in Asiala *et al* (1996), looks at interview transcripts in very fine detail. We try to find mathematical points as narrow as possible on which there is a range of student performance. Then we try to find explanations of the differences in terms of constructing or not constructing specific actions, processes, objects and/or schemas. In the totality of these local explanations, APOS Theory offers explanations of student successes and failures.

**Be applicable to a broad range of phenomena.** APOS Theory has been applied, both by its developers and by others, to a large number of undergraduate mathematics topics. For a partial list, see Dubinsky and McDonald (In review).

**Help organise thinking about learning phenomena.** Using APOS Theory to develop a genetic decomposition of a mathematical concept is one way of organising one's thinking about how students can learn the concept.

**Serve as a tool for analysing data.** A very specific method of using APOS Theory to analyse data was

mentioned above and is described in some detail in Asiala *et al* (1996).

**Provide a language for communication about learning.** Dubinsky and McDonald (In review) describe how a large number of researchers use the language of APOS Theory in their discussions. Terms such as action, process, object, schema, interiorization and encapsulation are now commonly used in discourse about learning and teaching collegiate mathematics.

### **What can happen when APOS Theory is used?**

The paradigm described in Figure 1 has been applied to courses in pre-calculus, calculus, discrete mathematics, and abstract algebra. Determining the results of these experiments is not easy because, as indicated in our basic hypothesis stated at the beginning of this essay, an individual only has a *tendency* to respond to mathematical problem situations in terms of mental constructions which he or she has made or can make. This makes the standard kind of "objective" testing instrument less than perfectly reliable. On the other hand, qualitative instruments are time-consuming and so sample sizes have to be small. The research studies connected with our use of the paradigm have tried to deal with this by using a synthesis of quantitative and qualitative methods as described in Asiala *et al* (1996). In some cases comparisons are made with students who took the same course with standard pedagogy. In other cases, only the results of using the pedagogy described here are reported. There is also data comparing student attitudes towards mathematics after experiencing our pedagogy and standard pedagogy. Finally, there are results on the long-term effects of using the approach described in this essay.

The results of 13 of these studies are summarised in Weller *et al* (In review). The conclusions stated in that summary are that the data is striking, consistent, and broad-based in favour of our approach. Specifically,

- The comparative data from the calculus and abstract algebra studies consistently point in the direction of the experimental group. In essentially every single category under consideration, students from the experimental groups achieved a higher level of performance than did students from the traditional groups.
- With respect to the non-comparative data, students who received instruction based upon APOS Theory and the ACE Teaching Cycle performed at a level complementary to and consistent with what was reported in the comparative analysis. Based upon our collective

teaching experience, we believe the non-comparative results reported in this paper would likely be higher than what one would typically expect from students having completed similar traditionally structured courses.

- Instruction based upon APOS Theory seems to generate student interest and enthusiasm for mathematics, as well as potential, sustained long-term academic benefits. Students who completed the experimental courses tended to focus attention upon their learning, as opposed to utilitarian aspects of coursework, such as performance on tests. At least in calculus, there does not appear to be a significant Hawthorne effect. Students who received APOS-based instruction performed at least as well as students who received traditional instruction in subsequent mathematics courses. In addition, students who completed APOS courses were more likely to pursue additional study of mathematics.

Given the wide variation of the topics covered in these papers and the consistency of the performance results, one could conclude that the direction of the results in favour of the experimental groups is probably not due to chance. These results suggest that instruction based

upon APOS Theory yields results that are better than what one would expect within a traditional setting. Moreover, these papers provide compelling evidence that the research framework based upon APOS Theory and the ACE Teaching Cycle may be a valid tool to describe and to enhance student learning of mathematics.

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## Linking research and teaching through ICT

University of Warwick, 7 February 2001

Meeting Report by: Chris Sangwin, University of Birmingham, C.J.Sangwin@bham.ac.uk

This short article reports on the recent meeting organized to disseminate the results of the TELRI — Technology Enhanced Learning in Research-led Institutions — project. This project has been exploring relationships between research and learning. That is, how to encourage students to develop research-like attitudes and learning strategies. While this is not specific to mathematics these techniques and approaches have been applied in this field.

### **Adoptive vs Adaptive learning**

In order to describe and differentiate between types of learning the TELRI Project have defined *Adoptive* and *Adaptive* patterns of behaviour. The former is essentially a reproductive mode of operation where students will learn verbatim facts, procedures and skills. In the latter students will develop general ideas, use insight and apply skills creatively to new situations. Each broad category has *deep* and *surface* aspects. For example, mastering a complex computer package might be classed as adoptive learning. The skills and techniques needed are bounded and well established, hence adoptive, however this is not “surface” learning as the tasks are

complex, inter-related and have meaning. However, the ‘meaning’ of the operations is restricted to the context of the computer package (i.e. example) and is therefore not transferable.

Clearly some adoptive learning is both desirable and necessary in any degree programme but “*it is believed that research-orientated learning is more likely to develop students’ abilities to transfer their learning processes into new situations, so that they can develop as experts rather than as competent practitioners.*” (This and all following unattributed quotes come from the TELRI staff pack.)

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Thus the TELRI approach focuses on “*embedding research-based approaches to learning into curricula through effective application of technology.*”

### **Encouraging adaptive learning**

If one accepts that “*there is a natural propensity to use previously formed concepts before developing new ones*”, one would expect students to use primarily adoptive approaches when these fulfill the requirements of any assessments. Thus students will be encouraged to learn adaptively only if this expectation is perceived through the assessments students are required to complete.

The TELRI (interim) guidelines for course design list the following characteristics of assignments that assess expertise:

- There is no unique, established solution or correct response.
- No purely procedural, ‘algorithmic’ or learned response will suffice.
- Judgments of value, likelihood and probability are required.
- Originality, innovation, creativity, insight, personal inspiration and reflection are required.
- Tasks are concerned primarily with the conceptualization ... as opposed to substantive knowledge.
- Scope within the task for creative knowledge.

To better explain and encourage the use of the techniques developed by TELRI a *Staff Pack* has been produced. This consists of; (i) an introductory leaflet explaining the principles of their approach; (ii) the booklet “Supporting high level learning through research-based methods”; and (iii) a selection of case studies, which unfortunately does not currently include mathematics.

### **Applying Research Based Approaches in Mathematics**

Certainly few mathematicians would accept that all these characteristics apply to mathematics. That there is a “correct answer” and often a “unique solution” is one aspect that makes mathematics unique. However, there is scope for creativity and skill in obtaining this solution and so one might argue that “solution” in this context means the method not the result. Encouraging students to be creative, but correct, in their thinking is one of the great teaching challenges. Of course an “expert” in mathematics possesses skills in correctly and efficiently performing procedures and algorithms. Given this and the observation that many, perhaps the majority, of courses in mathematics favour adoptive learning it is perhaps hard to imagine how they could be

modified to favour assignments with the above research based characteristics. Mathematics has a propensity to favour the adoptive mode, not unnaturally in a subject that has a highly developed sense of *truth* and *correctness*. One might be led to believe that scope for adaptive learning was strictly limited.

It is then not surprising that many students fail to display the the sorts of attitudes staff desire and expect. More worryingly they fail even to attain the status of competent practitioner. For example, there exist undergraduate students who think that a half plus a half equals  $1/(2+2)$  which equals one quarter. Of course, were one to present them with the results of their calculation in this rhetorical form they may well understand the evident fallacy. However, at the level of symbolic manipulation they happily carry out nonsensical operations with little or no appreciation of, or apparent interest in ascribing meaning to, the results they obtain. Furthermore, it is common to encounter students who display great difficulty in applying their knowledge to situations that are only marginally different from the examples to which they have been exposed. There are still others who do not correctly appreciate the scope of validity of a theorem or method they wish to apply. Surely it is precisely these sorts of attitudes that one would wish to change?

So exactly what is research and how would one undertake this at undergraduate level? Research literally means “*systematic investigation of materials, sources, etc. to establish facts*” (Oxford English Dictionary). Accepting in addition to this that, in the context of undergraduate students, there need be no requirement of novel content, the possibility of having students undertake research begins to make more sense. So what are “research attitudes” and how does one instil them? At one level this could simply mean asking questions such as “I wonder if ...” and “What happens if ...” - precisely the way more mainstream research is initiated. When students begin to ask such questions routinely they start to move beyond simply seeing the subject as a set of algorithmic tricks to be applied in special cases and begin to appreciate the consistent whole which makes the subject appealing. Certainly such attitudes are indicative of students who care about their work. When students do care they *want* to acquire the necessary knowledge and strive to do so. Perhaps effort expended in the design of assignments which require students to develop research attitudes to their thinking would be of great benefit.

### **Further information**

More information of the TELRI project may be found at <http://www.telri.ac.uk/>