RESEARCH STATEMENT

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1. Introduction

I am interested in algebraic combinatorics and representation theory, particularly in symmetric function theory.

Representation theory is the study of groups and algebras through the way they act on vector spaces, and symmetric functions play an important role in this theory, as they arise as characters of representations. Even in the seminal work of Frobenius [F00] and Young, combinatorics played a crucial part in their study of the symmetric group. Primary problems of interest in combinatorial representation theory include classifying irreducible representations, constructing irreducible modules, finding character formulas, and determining decompositions of modules into their irreducible components. In each case, the goal is to find solutions via combinatorial techniques, such as finding bijections with nice combinatorial objects [BR99].

My research (§2) has been focused on the combinatorics of Macdonald polynomials of general Lie type, using the connection with Cherednik's double affine Hecke algebras [C95]. There are interesting connections with spline theory (Macdonald polynomials as piecewise exponential functions) (§3), and I intend to continue my research in this area. I also aim to extend my research to investigate connections with geometry (intersection theory of affine Grassmannians) (§4), and to study representations of double affine Hecke algebras using combinatorial methods (§5).

2. Macdonald polynomials and alcove walks

Macdonald polynomials are multivariable Laurent polynomials depending on two parameters q and t that are associated to root systems, and they specialize to many well-known polynomials, such as Schur polynomials, Hall-Littlewood polynomials, Jack symmetric functions, and Koornwinder polynomials.

There are two kinds:

$$\text{nonsymmetric } E_\lambda(X; q, t), \text{ and symmetric } P_\lambda(X; q, t),$$

the latter of which can be obtained from the nonsymmetric version by symmetrization.

Macdonald [M87] originally defined this remarkable family of polynomials as eigenfunctions of a self-adjoint operator. In 1995, Cherednik [C95a] made a breakthrough in the study of these polynomials by showing how the intertwining operators of the double affine Hecke algebra can be used to generate nonsymmetric Macdonald polynomials. Another exciting

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development is the discovery by Haglund, Haiman and Loehr [HHL05], [HHL08] of combinatorial formulas involving tableaux for Type A Macdonald polynomials. In joint work with A. Ram [RY], we provided a formula for Macdonald polynomials for root systems of general Lie type. The result is a formula as a weighted sum of alcove walks, which are a discrete analogue of Littelmann paths [Li94].

Theorem 1. [RY]

$$E_\lambda = \sum_{p \in B(\lambda)} f_p X^{\mathtt{wt}(p)},$$

$$P_\lambda = \sum_{\substack{p \in B(\lambda) \\ \iota(p) \in W_0}} t^{\ell(\iota(p)w_0)/2} f_p X^{\mathtt{wt}(p)},$$

where $B(\lambda)$ is the set of folded alcove walks of type λ, and the coefficients

$$f_p = \prod_{\substack{k\text{th step} \\ \text{a + fold}}} \frac{t^{-1/2} - t^{1/2}}{1 - q^{\mathrm{sh}(p_k)} t^{\mathrm{ht}(p_k)}} \prod_{\substack{k\text{th step} \\ \text{a - fold}}} \frac{(t^{-1/2} - t^{1/2}) q^{\mathrm{sh}(p_k)} t^{\mathrm{ht}(p_k)}}{1 - q^{\mathrm{sh}(p_k)} t^{\mathrm{ht}(p_k)}}$$

are rational functions in q and t.

An application of the above work has already been found by Lenart [Le09], who was able to use Theorem 1 in the Type A symmetric case to derive a Haglund-Haiman-Loehr type formula.

Extending the above ideas, we can use the alcove walk model to compute products and prove a Littlewood-Richardson rule for Macdonald polynomials.

Theorem 2. [Y]

$$E_\lambda P_\mu = \sum_{p \in \Gamma(\lambda,\mu)} c_p E_{\mathtt{wt}(p)},$$

$$P_\lambda P_\mu = \sum_{p \in \Gamma(\lambda,\mu)} c_p e_p P_{\mathtt{wt}(p)},$$

where $\Gamma(\lambda, \mu)$ is the set of folded alcove walks of type λ^{-1}, beginning in the alcoves of the polytope centered at $-w_0\mu$, contained in the dominant chamber, with bi-coloured folds. The coefficients c_p and e_p are rational functions in q and t that depend on the shape of the folded walk p.

By setting $q = 0$, the second formula in Theorem 2 becomes the formula of Schwer [Sc06] for products of spherical functions in terms of positively folded galleries. By setting $q = t$, this becomes the classical Littlewood-Richardson rule for Weyl characters.

Some combinatorial questions. One of the most appealing aspects of the formulas of Haglund et al. and Lenart is that they involve fewer terms than the alcove walk formula. In the latter, this is achieved by a "compression" of terms which retains the combinatorial flavour.

Problem 1. *In what ways can the compression of alcove walks be achieved in other Lie types?*

For example, for the Type A_1 root system, Macdonald [M03] showed that there is a q-binomial formula for the symmetric polynomial $P_{k\alpha}$ with $2k+1$ terms (and this is optimal), while the alcove walk formula in Theorem 1 involves 2^k terms.

One way to tackle this problem is to use a case by case approach for the classical Lie types. Recent development towards answering this problem was made by Lenart [Le09a] in the case of Type B and C Hall-Littlewood polynomials. The idea is to use equivalence classes of walks to create fillings of tableaux. A variant of this idea can be used, where alcove walks of "similar shape" can be grouped together to form one path that represents the equivalence class.

Instead of a case by case approach, one can study compression for all classical Lie types at once by focusing on the nonreduced root system of Type $C^\vee C$, which gives rise to the Koorwinder polynomials. The affine Weyl group associated to this root system has five orbits, leading to Macdonald polynomials with six parameters, and at various specializations of these parameters, one can obtain the root systems of classical Lie type.

Problem 2. *Is there a "nice" combinatorial formula of the form*

$$E_\lambda E_\mu = \sum_p a_p E_{\mathrm{wt}(p)}$$

for nonsymmetric Macdonald polynomials?

It seems possible that the alcove walk methods have a chance of producing such a formula if we can control the combinatorics.

3. Connections with spline theory

In joint work with A. Ron, we showed that the Type A_1 and A_2 nonsymmetric Macdonald polynomials are piecewise exponential functions.

Problem 3. *Show that nonsymmetric Macdonald polynomials of general Lie type are piecewise exponential functions.*

A broader goal is to connect spline theory and representation theory via the development of a new class of multivariate splines $B_{\lambda,k}$, which we call modulation splines, that are a continuous analogue of Macdonald polynomials. These splines form a dual theory for the theory of box splines [BHK93], which has been long sought after by approximation theorists.

Inspired by the theory of Macdonald polynomials, modulation splines are constructed by the noncommutative operations of modulation and convolution. This departs from classical spline theory, where splines are usually constructed with commuting operators.

As a first step, we have the following problem:

Problem 4. *Find closed formulas for the Fourier transform of modulation splines. They should be of the form*

$$B_{\lambda,k} = \sum_{j=0}^{k-1} L_j \sum_{\mu \in W_0\lambda} (-1)^{\ell(\mu)} e_\mu,$$

$$B_{\lambda,k} = \sum_{j=-k-1}^{k-1} \sum_{\omega} M^{j\omega} I_{j\omega} \sum_{\mu \in W_0\lambda} (-1)^{\ell(\mu)} e_\mu,$$

where L_j is a product of $k|R_+|$ integrations and a modulation, and $I_{j\omega}$ is a sum of k modulations and an integration.

These formulas show that $B_{\lambda,k}$ are smooth compactly supported piecewise analytic functions. A solution to this problem would lead to a solution to Problem 3 via a process of discretization. So far, we have been successful in the Type A_1 case, and have obtained computational evidence for the existence of such formulas in the Type A_2 case. From a spline-theoretic point of view, it is the Type A_2 case that is the most important, as it has applications in 2-dimensional imaging.

We expect that this development will lead to new problems and techniques for both spline theory and Macdonald theory. One possible avenue is to investigate modulation splines that are not attached to a root system, but rather to an arbitrary multiset of vectors.

4. Connections with geometry

Classically, the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$ are intersection numbers for Schubert varieties of Grassmannians, so we can ask

Problem 5. *What is the analogue for the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}(q,t)$ for symmetric Macdonald polynomials?*

More generally, what we would really like to achieve is

Problem 6. *Find a geometric interpretation for the path combinatorics of the double affine Hecke algebra.*

In the paper [GL95], Gaussent and Littelmann connect the combinatorics of alcove walks to the geometry of the affine Grassmannian. One of their results show that positively folded alcove walks of the same type and the same endpoint naturally index the Mirković-Vilonen cycles, and hence gives geometric meaning to the spherical affine Hecke algebra. The paper by Parkinson, Ram and Schwer [PRS09] extends this work by showing that labelled positively folded alcove walks index points in the affine flag variety, which makes explicit the connection between alcove walks and the geometry of the affine Hecke algebra.

It is expected that the double loop group will play a similar role for the double affine Hecke algebra.

5. Representations of double affine Hecke algebras

Another direction I would like to delve into is the large body of work on representations of double affine Hecke algebras. Introduced by Cherednik, these algebras were used to prove Macdonald's conjectures [C95], and have connections with diverse areas of mathematics such as quantum groups and integrable systems.

In the paper [BEG03], Berest, Etingof and Ginzburg give a complete classification of finite dimensional irreducible representations of Type A rational Cherednik algebras, which are a certain degeneration of double affine Hecke algebras. In [Go03], Gordon constructed simple rational Cherednik algebra modules with dimension $(h+1)^n$, where h is the Coxeter number of the Weyl group and n is the rank of the root system. This led to a proof of Haiman's conjecture on the existence of certain quotient rings of the ring of diagonal coinvariants, from which interesting combinatorics arise. The number $(h+1)^n$ counts the regions in the Shi hyperplane arrangement, and in the Type A case, is the number of parking functions of length $n+1$. Moreover, the number of dominant regions is the Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$, and there is a close connection between the lattice of noncrossing partitions and the arrangement of the dominant regions.

Problem 7. *What is the connection between the combinatorics arising from coinvariant rings and simple modules of double affine Hecke algebras?*

Cherednik showed in [C03] that there is a q-analogue of Gordon's result for double affine Hecke algebras, and intertwining operators played an important role there, so we would like to unravel this combinatorial connection using alcove walk techniques.

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