# NOTES FOR SHI REGIONS AND BASES OF POLYNOMIAL REPRESENTATIONS
# UW REPRESENTATION THEORY SEMINAR
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MARTHA YIP
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WI 53706

Contents



## 1. Six Bijections

A basis of the type $A_2$ Gordon module has sixteen elements. A basis of the zonotopal algebra corresponding to the type $A_2$ Shi arrangement also has sixteen elements. There is a series of bijections which gives a connection between a basis of the Gordon module and a basis of the zonotopal algebra.

$$\begin{aligned} \text{basis of Gordon module} &\longleftrightarrow \text{regions of Shi arrangement} \\ &\longleftrightarrow \text{parking functions} \\ &\longleftrightarrow \text{spanning trees} \\ &\longleftrightarrow \text{bases of a set } X \\ &\longleftrightarrow \text{basis for the central space } \mathcal{P}(X) \end{aligned}$$

1.1. **From the Shi arrangement to parking functions.** The type $A_{n-1}$ Shi arrangement is a collection of hyperplanes

$$x_i - x_j = d \qquad \text{for } 1 \leq i < j \leq n, \quad d = 0, 1, \tag{1.1}$$

in $\mathbb{R}^n$.

*Date*: April 14, 2008.

The regions of the hyperplane arrangment are the connected components of $\mathbb{R}^n$ minus the union of the hyperplanes. The fundamental region $R_0$ is the unique region lying on the positive side of all hyperplanes and whose closure contains the origin.

Let $R$ be a region and let

$$d(R) = \text{number of hyperplanes separating } R_0 \text{ and } R. \tag{1.2}$$

A parking function of length $n$ is a sequence of $n$ integers whose rearrangement in increasing order $a_1 \le \cdots \le a_n$ satisfies $0 \le a_i \le i-1$. For example, $(0,\dots,0)$ is always a parking function.

For example, there are three parking functions of length two:

$$(0,0), (1,0), (0,1).$$

**Theorem 1.1.** *(Stanley) There is a bijection between the regions of the type $A_{n-1}$ Shi arrangement and parking functions of length $n$. Moreover, if $(a_1,\dots,a_n)$ is the parking function associated to $R$, then $d(R) = \sum_{i=1}^n a_i$.*

*Proof.* Let $\varepsilon_1,\dots,\varepsilon_n$ be the standard basis for $\mathbb{R}^n$. Thinking of parking functions as vectors in $\mathbb{R}^n$, we inductively assign parking functions to the regions. Begin by labelling the fundamental region by

$$\lambda(R_0) = (0,\dots,0).$$

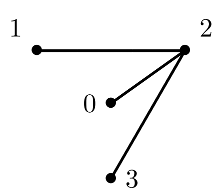
If $R$ is a region adjacent to $R_0$, set

$$\lambda(R) = \begin{cases} \lambda(R_0)+\varepsilon_j & \text{if } R \text{ and } R_0 \text{ are separated by } x_i - x_j = 0, \\ \lambda(R_0)+\varepsilon_i & \text{if } R \text{ and } R_0 \text{ are separated by } x_i - x_j = 1. \end{cases} \tag{1.3}$$

Repeat this process for the remaining regions adjacent to the labelled regions. This labelling is independent of the order which we assign labels, and only depends on the hyperplanes that separate the region from the fundamental region.

1.2. **From parking functions to spanning trees.** Let $T$ be a spanning tree on the vertex set $\{0,1,\dots,n\}$. For $i<j$, $i,j \ne 0$, the pair $(i,j)$ is an inversion of $T$ if the unique path in $T$ from 0 to $i$ contains $j$. Let $R(T)$ denote the set of inversions of $T$ and let $\ell(T) = |R(T)|$ be the number of inversions of $T$. Note that $\ell(T) \le \binom{n}{2}$.

For example,

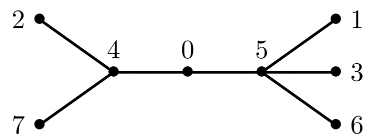
$T =$ 


$R(T) = \{(1,2)\}$

$\ell(T) = 1.$

**Theorem 1.2.** *(Kreweras) There is a bijection between spanning trees on the vertex set $\{0,1,\dots,n\}$ and parking functions of length $n$. Moreover, if the tree $T$ corresponds to the parking function $(a_1,\dots,a_n)$, then $\sum_{i=1}^n a_i = \binom{n}{2} - \ell(T)$.*

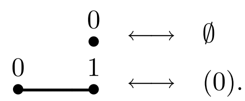
We give an example of the construction below. Let

$T =$ 

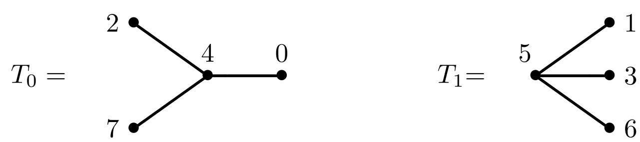

$R(T) = \{(1,5),(3,5),(2,4)\}$

$\ell(T) = 3.$

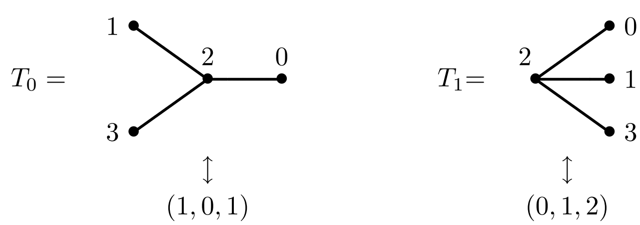
Begin by assigning




There is a unique path in $T$ from 0 to 1. Let $r$ be the vertex adjacent to 0 in this path, so $r = 5$. Delete the edge $(0, r)$ to decompose $T$ into two trees.




Relabel these trees, preserving the order of the labels. By induction, these trees correspond to parking functions




We want to associate a parking function $(a_1, \ldots, a_7)$ to $T$. Decomposing $T$ into two trees gives two parking functions with a total of six entries. The first entry in the parking function encodes information for putting the subtrees back together.

Let $g$ be the number of vertices in $T_1$ which are less than $r$, so $g = 2$ in this example. Set $a_1 = n - |T_0| - g = 1$.

Let $\{i_1, \ldots, i_k\}$ be the nonzero labels of $T_0$ listed in increasing order. If $(b_1, \ldots, b_k)$ is the parking function associated to $T_0$, then assign $a_{i_j} = b_j + |T_1|$ for $1 \le j \le k$. Let $(c_1, \ldots, c_{n-k-1})$ be the parking function associated to $T_1$. In the remaining entries of the parking function $(a_1, \ldots, a_n)$, fill with the entries of $(c_1, \ldots, c_{n-k-1})$ in order.

In this case, we have

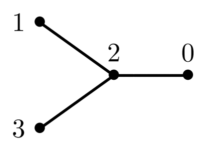
$$(a_1, \ldots, a_7) = (1, 1+4, 0, 0+4, 1, 2, 1+4) = (1, 5, 0, 4, 1, 2, 5).$$

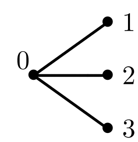
Note that $\sum_{i=1}^{7} a_i = 18 = \binom{7}{2} - \ell(T)$.

We can reverse this construction. Let $A = (1, 5, 0, 4, 1, 2, 5)$ be a parking function, and note that the first entry is $a_1 = 1$. Let $u$ be the largest value so that $A\backslash\{a_1\} \cup \{u\}$ remains a parking function. In this case, $u = 3$, since $A - a_1$ rearranged in increasing order is $0 \leq 1 \leq 2 \leq 4 \leq 5 \leq 5$. Observe that $u = n - |T_0|$, so $|T_1| = n + 1 - |T_0| = u + 1 = 4$. Decompose $A\backslash\{a_1\}$ into two ordered subsets by letting $A_0$ be the set of elements that are greater than or equal to 4, and letting $A_1$ be the set of elements less than 4. Subtract $|T_1|$ from every element of $A_0$, and we obtain two parking functions

$$A_0 = (a_2 - 4, a_4 - 4, a_7 - 4) = (1, 0, 1) \qquad \text{and} \qquad A_1 = (a_3, a_5, a_6) = (0, 1, 2).$$

They inductively correspond to two trees $T_0$ and $T_1$.

$(1, 0, 1) \quad \longleftrightarrow$ 


$(0, 1, 2) \quad \longleftrightarrow$ 


Reassign the nonzero labels to $T_0$ in increasing order as determined by the indices of the elements in $A_0$. To recover the root vertex $r$ in $T_1$, let $l = u - a_1 + 1 = 3$. Choose the $l$th smallest index in $A_1 \cup \{a_1\}$ and let that be $r$. In this case, the indices in $A_1 \cup \{a_1\}$ are $\{1, 3, 5, 6\}$, so the third smallest index is $r = 5$. Reassign $r$ to the zero vertex in $T_1$, and reassign the remaining labels to $T_1$ in increasing order as determined by the indices of the elements in $A_1 \cup \{a_1\}\backslash\{r\}$.

Lastly, join the edge $(0, r)$ to obtain the tree $T$.

1.3. **From spanning trees to bases of $X$ to a basis for $\mathcal{P}(X)$.** We shall focus on the case $n = 3$. Let

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3 \ \alpha_{12} \ \alpha_{13} \ \alpha_{23}] \tag{1.4}$$

be an ordered set. Here, $\alpha_{ij} = \varepsilon_i - \varepsilon_j$.

The zonotope of $X$ is

$$Z(X) = \left\{ \sum_{x \in X} t_x x \mid 0 \leq t_x \leq 1 \right\}. \tag{1.5}$$

The central space of the zonotope is

$$\mathcal{P}(X) = \operatorname{span}_{\mathbb{R}}\{p_Y \mid \operatorname{span}(X - Y) = \mathbb{R}^n\}, \tag{1.6}$$

where

$$p_y = y_1 x_1 + \cdots + y_n x_n \qquad \text{and} \qquad p_Y = \prod_{y \in Y} p_y. \tag{1.7}$$

The set of bases of $X$ is

$$\mathbb{B} = \{B \subseteq X \mid B \text{ is a basis for } \mathbb{R}^n\}. \tag{1.8}$$

**Lemma 1.3.** *There is a bijection between spanning trees on the vertex set $\{0, 1, \ldots, n\}$ and bases of $X = \{\varepsilon_i \mid 1 \leq i \leq n\} \cup \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$.*

*Proof.* Given a tree $T$, construct a corresponding set of vectors $B_T \subseteq X$ by letting $\varepsilon_i \in B_T$ whenever the edge $(0,i)$ is in $T$, and letting $\alpha_{ij} \in B_T$ whenever the edge $(i,j)$ is in $T$. It is clear that $B_T$ contains $n$ vectors. The fact that the vectors are linearly independent follows because the edges of $T$ do not form any cycles. □

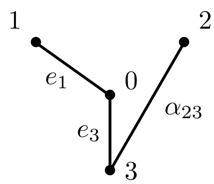
For $B \subseteq X$, define

$$X(B) = \{x \in X \mid x \notin \operatorname{span}\{b \in B \mid b \leq x\}\}. \tag{1.9}$$

In particular, note that $X(B) \cap B = \emptyset$, so $X - X(B)$ spans $\mathbb{R}^n$, and so $p_{X(B)} \in \mathcal{P}(X)$.

**Theorem 1.4.** *(Dyn, Ron) The set* $\{p_{X(B)} \mid B \in \mathbb{B}(X)\}$ *is a basis for* $\mathcal{P}(X)$.

An example.

$T =$ 


$$\longleftrightarrow \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \longleftrightarrow \quad \begin{array}{c} p_{X(B)} = x_2(x_1 - x_2) \\ X(B) = \{e_2, \alpha_{12}\} \end{array}$$

1.4. **From the Shi arrangement to a basis for the Gordon module.** Again, we focus on the case $n = 3$. Let $\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be the weight lattice, and let $\mathfrak{h}^* = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}$. Let

$$\rho_c = \tfrac{4}{3}(\omega_1 + \omega_2) \in \mathfrak{h}^*. \tag{1.10}$$

The element $s_0, s_1, s_2 \in W$ act as reflections on $\mathfrak{h}^*$. In particular, they act on $\rho_c$ and send $\rho_c$ to different alcoves. Each alcove has a (minimal length) sequence of reflections taking $\rho_c$ to that alcove.

As a $\mathbb{C}[q^{\pm 1/3}]$-algebra, the double affine Hecke algebra of type $A_2$ has generators $T_1, T_2, g, X^{\omega_1}, X^{\omega_2}$ with relations

$$\begin{gathered}
T_1T_2T_1 = T_2T_1T_2, \\
T_1X^{\omega_1}T_1 = X^{\omega_1-\omega_2}, \quad T_1X^{\omega_2} = X^{\omega_2}T_1, \quad T_2X^{\omega_1} = X^{\omega_1}T_2, \quad T_2X^{\omega_2}T_2 = X^{-\omega_1+\omega_2}, \\
g^3 = 1, \quad gT_1 = T_2g, \quad gX^{\omega_1} = q^{1/3}X^{-\omega_1+\omega_2}g, \quad gX^{\omega_2} = q^{2/3}X^{-\omega_1}g, \quad t = q^c, \\
T_i = (t^{1/2} - t^{-1/2})T_i + 1.
\end{gathered}$$

The polynomial representation $M(\mathbf{1}) = \mathbb{C}[X]\mathbf{1}$ is given by

$$T_i\mathbf{1} = t^{1/2}\mathbf{1} \qquad \text{and} \qquad g\mathbf{1} = \mathbf{1}. \tag{1.11}$$

If we set $c = 4/3$, we get the simple quotient

$$L(\mathbf{1}) = M(\mathbf{1})/\text{unique maximal proper submodule}. \tag{1.12}$$

Define

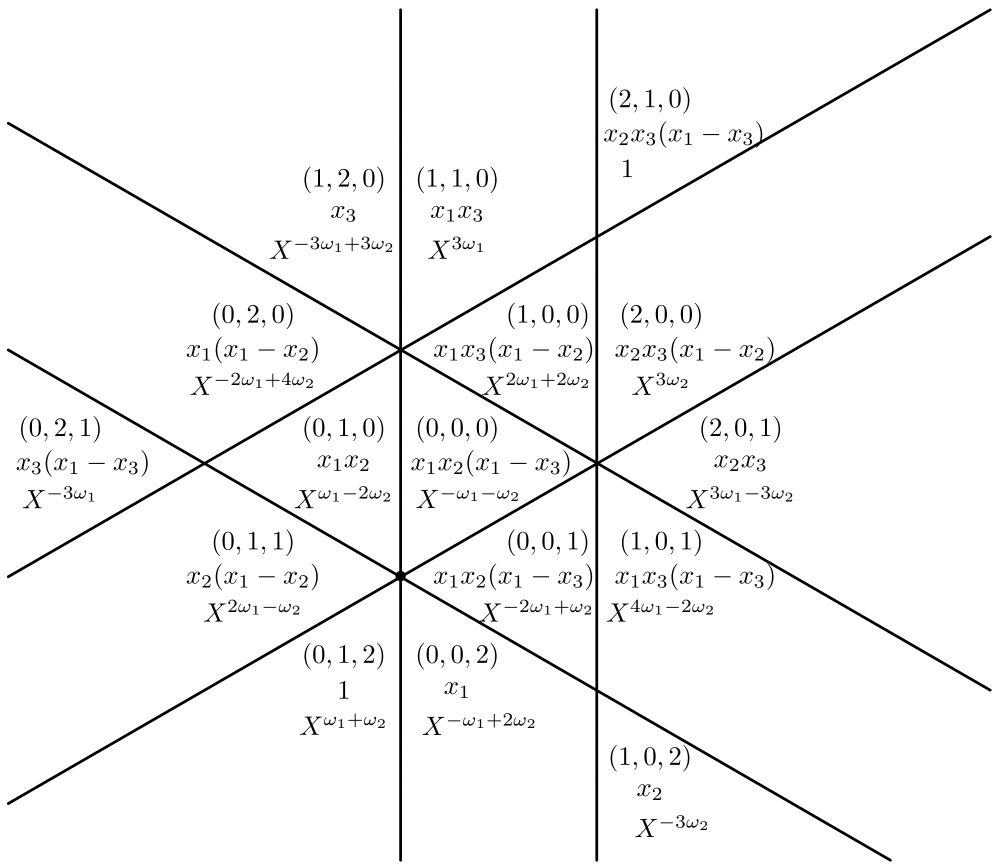
$$\begin{aligned}
T_0^\vee &= (X^{\omega_1+\omega_2}T_2T_1T_2)^{-1}, \qquad T_1^\vee = T_1, \qquad T_2^\vee = T_2, \\
Y^{\omega_1^\vee} &= gT_2T_1, \qquad Y^{\omega_2^\vee} = g^2T_1T_2, \\
\tau_i^\vee &= T_i^\vee - (t^{1/2} - t^{-1/2})\frac{1}{1 - Y^{-\alpha_i^\vee}}.
\end{aligned}$$

**Theorem 1.5.** *The set*

$$\{\tau_{i_1}^\vee \cdots \tau_{i_k}^\vee \mathbf{1} \mid s_{i_1} \cdots s_{i_k}\rho_c \text{ indexes a minimal alcove of a region of the Shi arrangement}\}$$

*is a basis for* $L(\mathbf{1})$.

The following picture illustrates the correspondence between the basis elements of the Gordon module, and the basis elements of $\mathcal{P}(X)$, together with the corresponding parking function.

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