Combinatorics of Shi arrangements

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Preliminaries

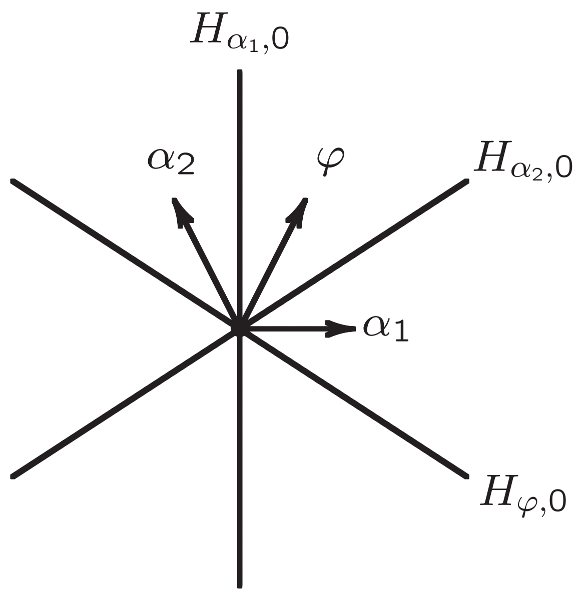
Let $\mathfrak{h}^*$ be a finite dimensional $\mathbb{R}$ vector space.

Let $GL(\mathfrak{h}^*)$ be the group of invertible linear transformations of $\mathfrak{h}^*$.

A Weyl group $W \leq GL(\mathfrak{h}^*)$ is a finite reflection group acting on a lattice $P \subseteq \mathfrak{h}^*$, such that $\mathfrak{h}^* = P \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $R^+ \subseteq \mathfrak{h}^*$ be an index set for the reflections in W.

The Coxeter arrangement for $W = A_2$.



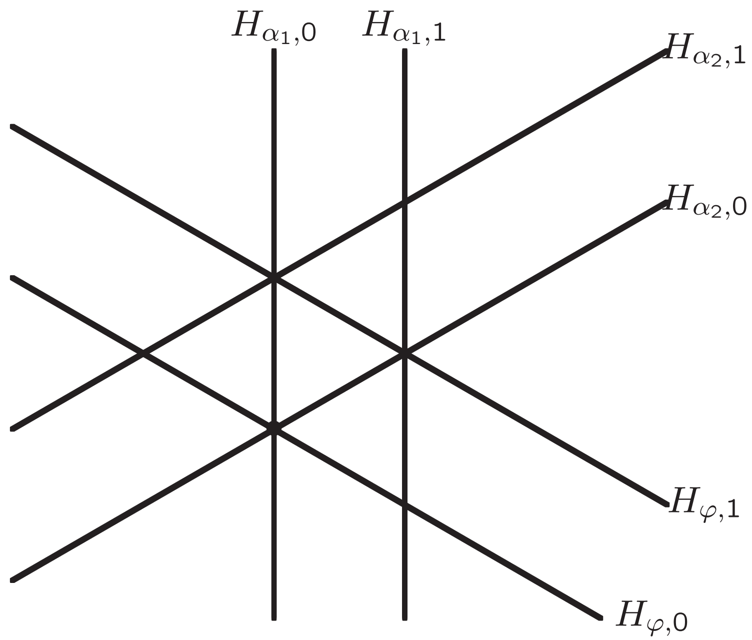
$R^+ = \{\alpha_1, \alpha_2, \varphi\}$

$W \cong S_3$ is generated by reflections s_1, s_2 in the hyperplanes $H_{\alpha_1,0}, H_{\alpha_2,0}$.

The Shi arrangement is the collection of hyperplanes

$$\{H_{\alpha,i} \mid \alpha \in R^{+}, i = 0, 1\}.$$

The A_2 Shi arrangement:



Let

r be the rank of the arrangement,

h be the Coxeter number of W,

$e_1, \ldots, e_r$ be the exponents of W.

- Theorem (Shi):

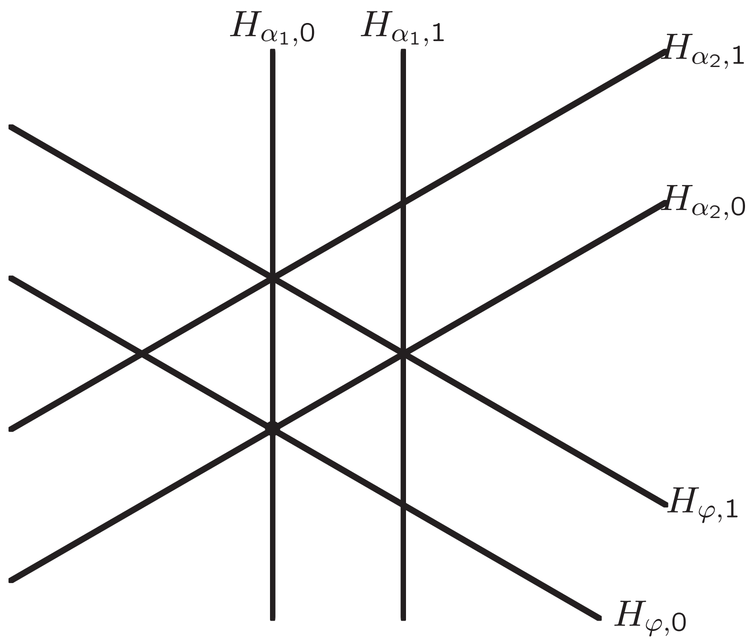
$$\#\text{ of regions in the Shi arrangement } = (h+1)^r$$

- Theorem (Athanasiadis):

$$\#\text{ of dominant regions } = \frac{1}{|W|} \prod_{i=1}^{r} e_i + h + 1$$

$$\#\text{ of bounded dom. regions } = \frac{1}{|W|} \prod_{i=1}^{r} e_i + h - 1$$

For $W = A_2$:



rank $= 2$
Coxeter number $h = 3$
exponents $e_1, e_2 = 1, 2$

# of regions $= 4^2 = 16$

# of dominant regions $= \frac{1}{6}(5 \cdot 6) = 5$

# of bounded dom. regs. $= \frac{1}{6}(3 \cdot 4) = 2$

Type A enumerative connections

A parking function of length n is a sequence of integers such that when arranged in increasing order $a_1 \leq \ldots \leq a_n$, it satisfies

$$0 \leq a_i \leq i - 1.$$

When $n = 2$, the parking functions are $(0,0),(1,0),(0,1)$

How many are there?

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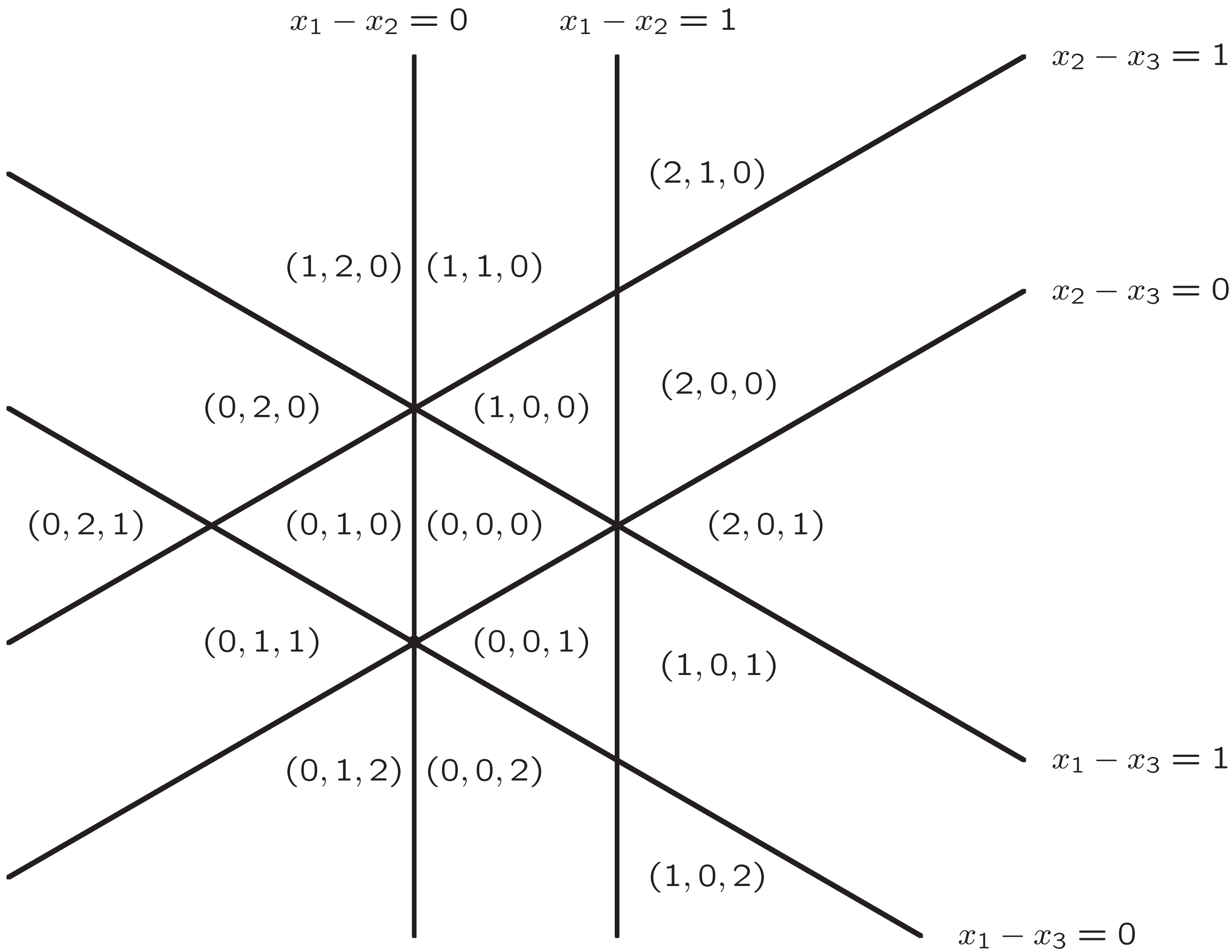
Theorem (Stanley): There is a bijection

$$\{\text{regions of Shi arr.}\} \longleftrightarrow \{\text{parking functions}\}$$


$x_1 - x_2 = 0$
$x_1 - x_2 = 1$
$x_2 - x_3 = 1$
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$x_1 - x_3 = 1$
$x_1 - x_3 = 0$
$(2,1,0)$
$(1,2,0)$
$(1,1,0)$
$(2,0,0)$
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For $W = A_{n-1} \cong S_n$,

Coxeter number is $h = n$, and rank $= n - 1$,

so $(h+1)^r = (n+1)^{n-1}$.

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Theorem (Kreweras): There is a bijection

$$\{\text{parking functions}\} \longleftrightarrow \{\text{spanning trees on } n + 1 \text{ vxs.}\}$$

Related statistics

$$\{\text{Shi regions}\} \leftrightarrow \{\text{parking functions}\} \leftrightarrow \{\text{spanning trees}\}$$

$$R \quad \leftrightarrow \quad a_1 \le \cdots \le a_n \quad \leftrightarrow \quad T$$

$$d(R) \quad = \quad \sum a_i \quad = \quad \binom{n}{2} - \mathrm{inv}\,(T)$$

Connections with approximation theory

Let

$$X = \left\{ \begin{matrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{matrix} \right\} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha_{12}, \alpha_{13}, \alpha_{23}\}.$$

Given $y \in X$ and $Y \subseteq X$, let

$$p_y = y_1 x_1 + \cdots + y_n x_n \qquad \text{and} \qquad p_Y = \prod_{y \in Y} p_y.$$

For example, given $Y = \{\varepsilon_2, \alpha_{12}\}$, let

$$p_{\varepsilon_2} = x_2, \quad p_{\alpha_{12}} = x_1 - x_2, \quad \text{and} \quad p_Y = x_2(x_1 - x_2).$$

The central space of the zonotope of X is

$$\mathcal{P}(X) = \text{span}_{\mathbb{R}}\{p_Y \mid \text{span}(X - Y) = \mathbb{R}^n\}$$

Let

$$\mathbb{B}(X) = \{B \subseteq X \mid B \text{ is a basis for } \mathbb{R}^n\}.$$

Given $B \subseteq X$, define

$$X(B) = \{x \in X \mid x \notin \operatorname{span}\{b \in B \mid b \leq x\}\}.$$

Theorem (Dyn, Ron): A basis for $\mathcal{P}(X)$ is

$$\{p_{X(B)} \mid B \in \mathbb{B}(X)\}.$$

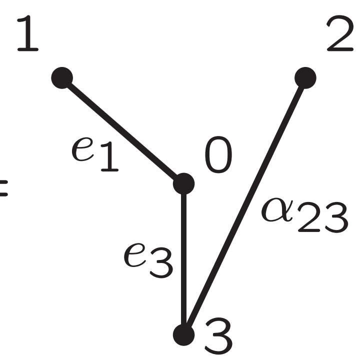
Lemma: There is a bijection

$$\{\text{spanning trees on } n+1 \text{ vxs.}\} \longleftrightarrow \mathbb{B}(X)$$

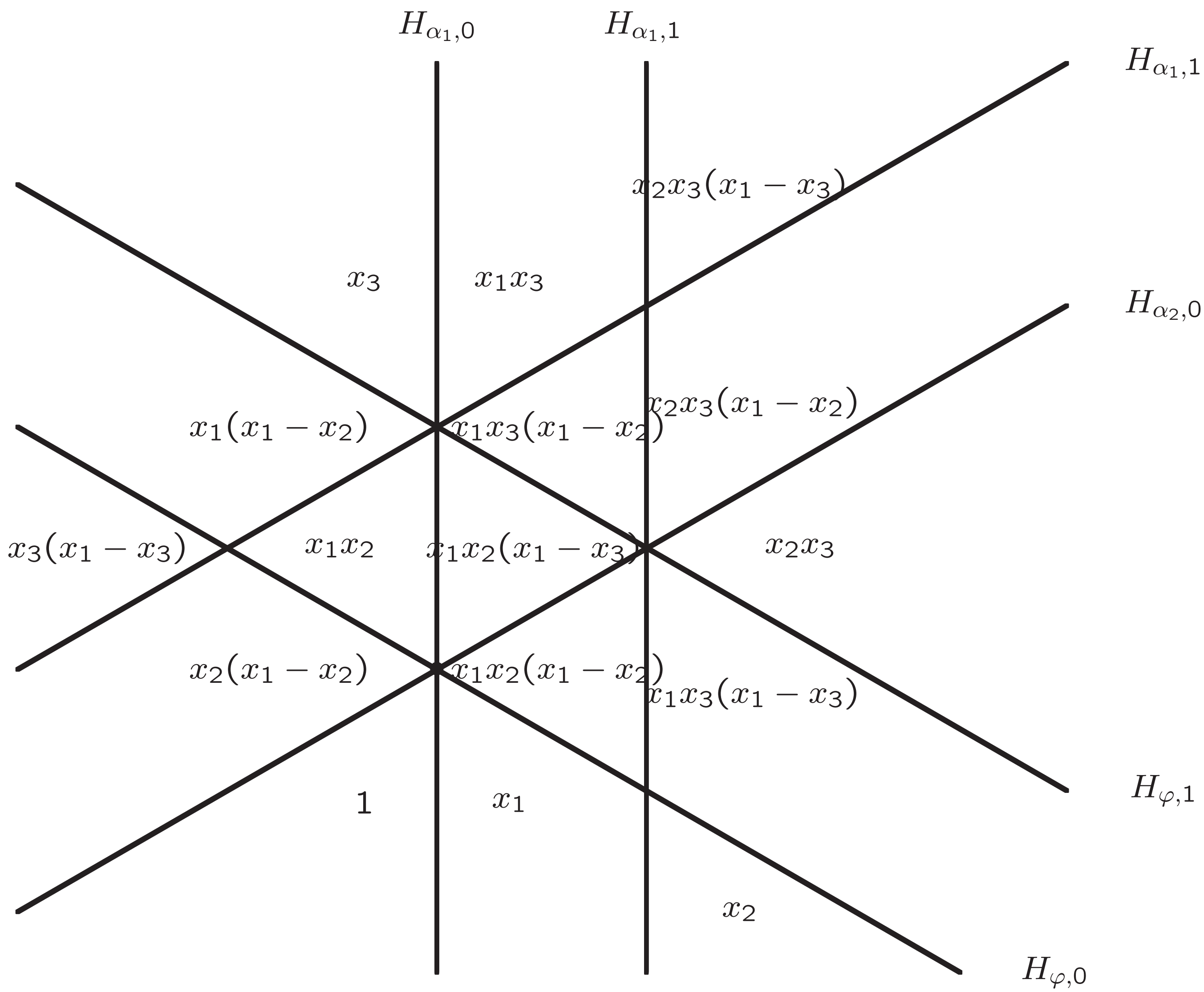
The regions of the Shi arrangement is an index set for a basis of the polynomial space $\mathcal{P}(X)$.

An example.

$T =$



$$\leftrightarrow \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \leftrightarrow \quad \begin{array}{c} p_{X(B)} = x_2(x_1 - x_2) \\ X(B) = \{e_2, \alpha_{12}\} \end{array}$$



$H_{\alpha_1,0}$
$H_{\alpha_1,1}$
$H_{\alpha_1,1}$
$x_2x_3(x_1-x_3)$
x_3
x_1x_3
$H_{\alpha_2,0}$
$x_2x_3(x_1-x_2)$
$x_1(x_1-x_2)$
$x_1x_3(x_1-x_2)$
$x_3(x_1-x_3)$
x_1x_2
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x_2x_3
$x_2(x_1-x_2)$
$x_1x_2(x_1-x_2)$
$x_1x_3(x_1-x_3)$
$H_{\varphi,1}$
1
x_1
x_2
$H_{\varphi,0}$

take a deep breath, just one more bijection to go...

Connections with another polynomial space

The double affine Hecke algebra $\widetilde{H}$ has basis

$$\{q^k X^\mu T_w Y^{\lambda^\vee} \mid k \in \tfrac{1}{m}\mathbb{Z}, \mu \in P, w \in W, \lambda^\vee \in P^\vee\},$$

and a bunch of relations.

Let $t = q^c$. Let

$$\rho_c = \frac{1}{2} c \sum_{\alpha \in R^+} \alpha.$$

The polynomial representation $M(\mathbf{1})$ of $\widetilde{H}$ is given by

$$T_i \mathbf{1} = t^{1/2}\mathbf{1}, \quad Y^{\lambda^\vee}\mathbf{1} = q^{\langle \lambda^\vee, \rho_c \rangle}\mathbf{1}, \quad g\mathbf{1} = \mathbf{1}.$$

Let

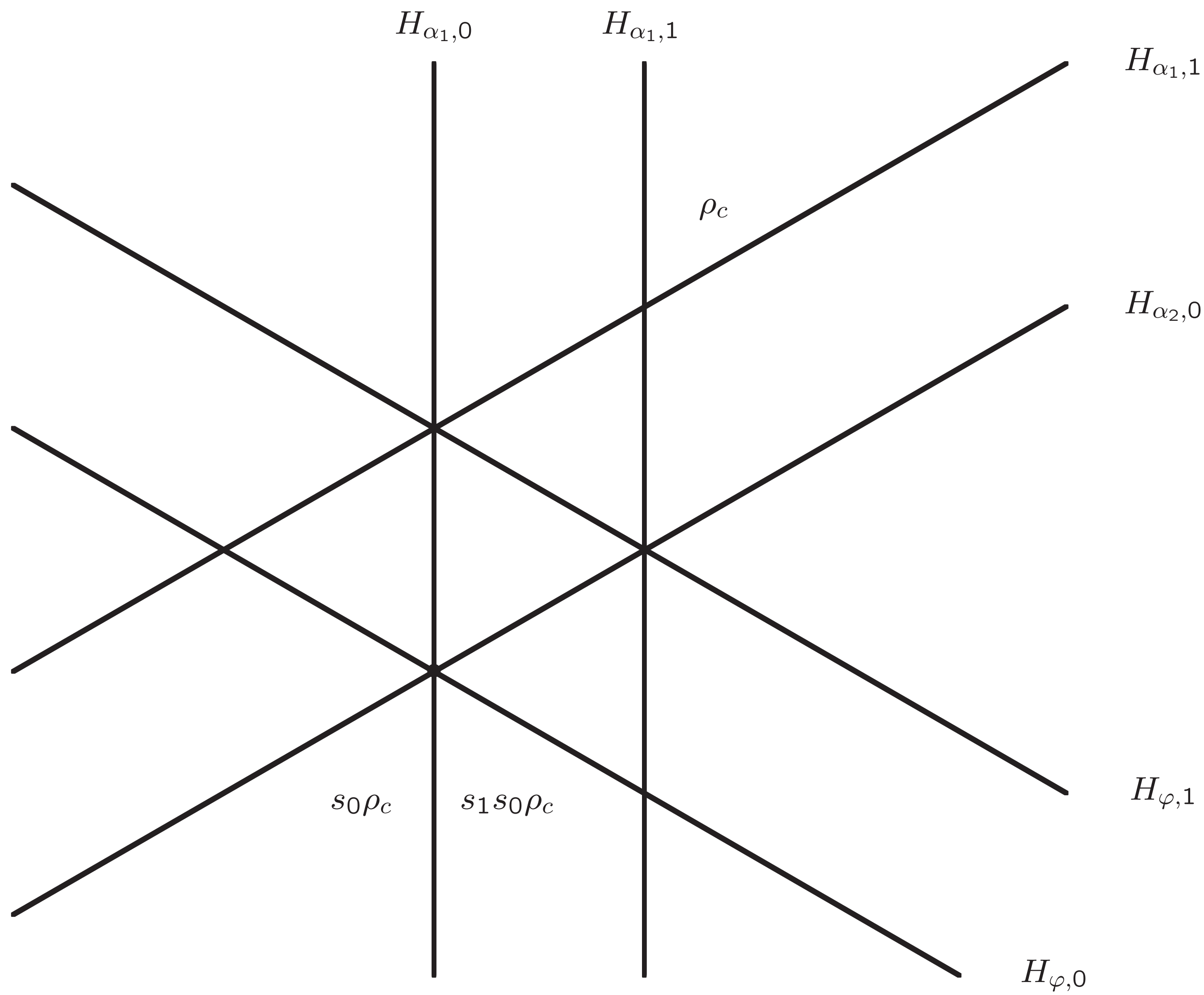
$$\tau_i^\vee = T_i^\vee - (t^{1/2} - t^{-1/2})\frac{1}{1 - Y^{-\alpha_i^\vee}}.$$

Fix $c = \frac{1}{h} + 1$, and let

$$L(\mathbf{1}) = M(\mathbf{1})/\text{unique maximal submodule.}$$

Theorem(Cherednik): A basis for $L(\mathbf{1})$ is

$$\{\tau_{i_1}^\vee \cdots \tau_{i_\ell}^\vee \mathbf{1} \mid s_{i_1} \cdots s_{i_\ell} \rho_c \text{ indexes a min. alcove of a Shi region }\}.$$


$H_{\alpha_1,0}$
$H_{\alpha_1,1}$
$H_{\alpha_1,1}$
ρ_c
$H_{\alpha_2,0}$
$s_0\rho_c$
$s_1s_0\rho_c$
$H_{\varphi,1}$
$H_{\varphi,0}$

What about the dominant regions?

In Type A_{n-1},

$$\frac{1}{|W|}\prod_{i=1}^{r} e_i + h + 1 = \frac{1}{n!}(n+2)(n+3)\cdots(2n) = \frac{1}{n+1}\binom{2n}{n}$$

$$\frac{1}{|W|}\prod_{i=1}^{r} e_i + h - 1 = \frac{1}{n!}(n)(n+1)\cdots(2n-2) = \frac{1}{n}\binom{2(n-1)}{n-1}$$

are both Catalan numbers.

There should be interesting analogues.