AFFINE BRAID GROUPS
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0.1. **Affine braid group.** Everything that follows is taken from [1]. Recall the extended affine Weyl group is

$$W \cong W_0 \ltimes t(L') \cong W_S \rtimes \Omega.$$

The *braid group* $\mathcal{B}$ of W is the group with generators $\{T(w) : w \in W\}$ and relations

$$T(v)T(w) = T(vw) \qquad \text{if } \ell(v) + \ell(w) = \ell(vw). \tag{0.1}$$

There is a canonical map $\mathcal{B} \to W : T(w) \mapsto w$.

Recall $s_0, \ldots, s_n$ generates W_S and $\Omega = \{u_j : j \in J\}$ is the group of length zero elements in W. Some facts about Ω that we need:

- The elements $u_j = u_{\pi'_j} = t(\pi'_j)v_{\pi'_j}^{-1}$ permute the simple affine roots a_i because $\ell(u_j) = 0$.
- They satisfy the property $u_j(a_0) = a_j$.
- Define $i + j \in J$ by $u_{i+j} = u_i u_j$.
- Almost always, Ω is a cyclic group. The only exception is the case D_{2n}, when $\Omega \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Notation: $T_i = T(s_i), U_j = T(u_j)$ for $0 \leq i \leq n$, $j \in J$. Here is an alternate presentation for $\mathcal{B}$.

Proposition 0.1. *The braid group $\mathcal{B}$ is generated by $T_0, \ldots, T_n$, U_j for $j \in J$, subject to the relations*

$$(T_iT_j)^{m_{ij}} = 1 \qquad \textit{if } (s_is_j)^{m_{ij}} = 1, \tag{0.2}$$

$$U_jU_k = U_{j+k} \qquad \textit{if } u_ju_k = u_{j+k}, \tag{0.3}$$

$$U_jT_iU_j^{-1} = T_{i+j} \qquad \textit{if } u_j(a_i) = a_{i+j}. \tag{0.4}$$

It's clear that $T_0, \ldots, T_n$, U_j for $j \in J$ generates $\mathcal{B}$ because each $w \in W$ can be written as $u_js_{i_1} \ldots s_{i_p}$, so $T(w) = U_jT_{i_1} \cdots T_{i_p}$.

Assume we have the relation (0.1). We get (0.2) because $(s_is_j)^{m_{ij}}$ is a reduced expression, we get (0.3) because $\ell(u_j) = 0$, and we get (0.4) because $u_js_i = s_{i+j}u_j$ is an element of length 1, so $U_jT_i = T(u_js_i) = T(s_{i+j}u_j) = T_{i+1}U_j$.

To motivate the construction of the double braid group, here is another presentation for $\mathcal{B}$. For $\lambda' \in L'$ dominant, define

$$Y^{\lambda'} = T(t(\lambda')). \tag{0.5}$$

Notice if $\lambda', \mu' \in L'$ are dominant, then[1] $\ell(t(\lambda'+\mu')) = \ell(t(\lambda')) + \ell(t(\mu'))$. So from relation (0.1), we get

$$\begin{aligned} Y^{\lambda'+\mu'} &= T(t(\lambda'+\mu')) = T(t(\lambda'))T(t(\mu')) = Y^{\lambda'}Y^{\mu'} \\ &= T(t(\mu'+\lambda')) = T(t(\mu'))T(t(\lambda')) = Y^{\mu'}Y^{\lambda'}. \end{aligned}$$

And if ν' is any element of L', it can be written as the difference of two dominant weights $\nu' = \lambda' - \mu'$. Define

$$Y^{\nu'} = Y^{\lambda'}(Y^{\mu'})^{-1}. \tag{0.6}$$

This is well-defined because if $\nu' = \lambda'_1 - \mu'_1 = \lambda'_2 - \mu'_2$, then $\lambda'_1 + \mu'_2 = \lambda'_2 + \mu'_1$, so $Y^{\lambda'_1}Y^{\mu'_2} = Y^{\lambda'_2}Y^{\mu'_1}$ implies $Y^{\lambda'_1}(Y^{\mu'_1})^{-1} = Y^{\lambda'_2}(Y^{\mu'_2})^{-1}$.

Hence $Y^{L'} = \{Y^{\lambda'} : \lambda' \in L'\}$ is an abelian subgroup of $\mathcal{B}$, isomorphic to L'. Under the canonical map $\mathcal{B} \to W : T(w) \mapsto w$, $Y^{L'}$ maps onto $t(L')$.

Let $\mathcal{B}_0$ denote the subgroup of $\mathcal{B}$ generated by $T_1, \ldots, T_n$.

Proposition 0.2. *The braid group $\mathcal{B}$ is generated by $\mathcal{B}_0$ and $Y^{L'}$, subject to the relations*

$$T_i^{-1}Y^{\lambda'} = Y^{\lambda'}T_i^{-1}, \qquad \text{if } \langle \lambda', \alpha_i \rangle = 0, \tag{0.7}$$

$$T_i^{-1}Y^{\lambda'}T_i^{-1} = Y^{s_i\lambda'}, \qquad \text{if } \langle \lambda', \alpha_i \rangle = 1, \tag{0.8}$$

for $1 \le i \le n$.

See that $T_0 = Y^{\varphi^\vee}T(s_\varphi)^{-1}$, and $U_j = Y^{\pi'_j}T(v_j)^{-1}$, since $s_0 s_\varphi = t(\varphi^\vee)$, and $u_j = t(\pi'_j)v_j^{-1}$ in W are reduced expressions. By Proposition 0.1, this shows that our set generates $\mathcal{B}$.

0.2. **Double affine braid group.** The double braid group was introduced by Cherednik. We saw that $\mathcal{B}$ is generated by $\mathcal{B}_0$ and $Y^{L'}$. We repeat this construction with the dual lattice L.

Here is the set-up. We have a pair of finite root systems (R, R') with the same associated finite Weyl group W_0, a pair of lattices (L, L') in a vector space E, and a pair of irreducible affine root systems (S, S') obtained from R and R'.

Macdonald considers three possibilities for these pairs:

(1) $(R, R^\vee)$, $(P, P^\vee)$, $(S(R), S(R^\vee))$
(2) (R, R), $(P^\vee, P^\vee)$, $(S(R)^\vee, S(R)^\vee)$
(3) (R, R) is of type C_n, $(Q, Q^\vee)$, $(S(R^\vee), S(R))$.

We have $\langle L, L' \rangle = e^{-1}\mathbb{Z}$ where e is the smallest integer such that $u^e = 1$ for every $u \in \Omega$, except in the case B_n or C_{2n} where $e = 1$. We have a pair of extended affine Weyl groups

$$W = W_0 \ltimes t(L') \qquad W' = W_0 \ltimes t(L).$$

Let $\Lambda = L \oplus e^{-1}\mathbb{Z}c$. The elements of Λ are affine-linear functionals on E, so elements look like $f = \lambda + rc$ where λ is some linear combination of weights ω_i, $r \in e^{-1}\mathbb{Z}$, and c is the constant function taking everything to 1. This is a lattice in $F = E^* \oplus \mathbb{R}$. We check that Λ is invariant under the action of W. Let $w = st(\mu') \in W$ and let $f = \lambda + rc \in \Lambda$, where $r \in e^{-1}\mathbb{Z}$. Then for $x \in E$,

$$\begin{aligned} (w \cdot f)(x) &= f(w^{-1} \cdot x) = f(t(-\mu')s^{-1}x) = f(s^{-1}x - \mu') = \langle \lambda, s^{-1}x - \mu' \rangle + r \\ &= \langle s\lambda, x \rangle + r - \langle \lambda, \mu' \rangle. \end{aligned}$$

[1] For any β', $\ell(t(\beta')) = \sum_{\alpha \in R^+} \langle \beta'_+, \alpha \rangle$.

Thus $wf = s\lambda + (r - \langle \lambda, \mu' \rangle)c$.

Let

$$X^{\Lambda} = \{X^f : f \in \Lambda\}, \tag{0.9}$$

be a multiplicative group isomorphic to Λ, so that $X^f X^g = X^{f+g} = X^g X^f$, and $(X^f)^{-1} = X^{-f}$

The *double braid group* $\widetilde{\mathcal{B}}$ is generated by $\mathcal{B}$ and X^{Λ}, subject to the relations

$$T_i X^f = X^f T_i, \qquad \text{if } \langle f, \alpha_i' \rangle = 0, \tag{0.10}$$

$$T_i X^f T_i = X^{s_i f}, \qquad \text{if } \langle f, \alpha_i' \rangle = 1, \tag{0.11}$$

$$U_j X^f U_j^{-1} = X^{u_j f}, \qquad \text{for all } j \in J, f \in \Lambda, \tag{0.12}$$

for $0 \le i \le n$. Recall the action of W on F is given by $s_i(\mu + rc) = s_i\mu + rc$, $t(\lambda')(\mu + rc) = \mu + (r - \langle \mu, \lambda' \rangle)c$.

Let $q = X^c$. We can show that q commutes with T_i and U_j, so it is a central element in $\widetilde{\mathcal{B}}$. Let $X^L = \{X^\lambda : \lambda \in L\}$.

Proposition 0.3. *The double braid group $\widetilde{\mathcal{B}}$ is generated by $\mathcal{B}_0$, X^L, $Y^{L'}$, and $q^{1/e}$, subject to the relations*

$$T_i Y^{-\lambda'} = Y^{-\lambda'} T_i, \qquad \textit{if } \langle \lambda', \alpha_i \rangle = 0, \tag{0.13}$$

$$T_i Y^{-\lambda'} T_i = Y^{-s_i \lambda'}, \qquad \textit{if } \langle \lambda', \alpha_i \rangle = 1, \tag{0.14}$$

$$T_i X^\lambda = X^\lambda T_i, \qquad \textit{if } \langle \lambda, \alpha_i' \rangle = 0, \tag{0.15}$$

$$T_i X^\lambda T_i = X^{s_i \lambda}, \qquad \textit{if } \langle \lambda, \alpha_i' \rangle = 1, \tag{0.16}$$

$$T_0 X^\lambda = X^\lambda T_0 \qquad \textit{if } \langle \lambda, \varphi' \rangle = 0, \tag{0.17}$$

$$T_0 X^\lambda T_0 = q^{-1} X^{s_\varphi \lambda} \qquad \textit{if } \langle \lambda, \varphi' \rangle = -1, \tag{0.18}$$

$$U_j X^\lambda U_j^{-1} = q^{-\langle \lambda, v_j \pi_j' \rangle} X^{v_j^{-1} \lambda}, \qquad \textit{for all } j \in J, \lambda \in L. \tag{0.19}$$

Theorem 0.4. *(Duality) Let $\widetilde{\mathcal{B}}'$ denote the group generated by $\mathcal{B}_0$, $X^{L'}$, Y^L, and $q^{1/e}$. There is an antiisomorphism $\omega : \widetilde{\mathcal{B}}' \to \widetilde{\mathcal{B}}$ such that $X^{\lambda'} \mapsto Y^{-\lambda'}$, $Y^\lambda \mapsto X^{-\lambda}$, $T_i \mapsto T_i$ for $i \neq 0$, and $q \mapsto q$.*

In $\widetilde{\mathcal{B}}$, $T_0 = Y^{\varphi^\vee} T(s_\varphi)^{-1}$, and $U_j = Y^{\pi_j'} T(v(\pi_j'))^{-1}$.

Let $\psi \in R$ be such that $\psi' \in R'$ is the highest root. So in the case $R' = R$, $\psi = \phi$, and in the case $R' = R^\vee \neq R$, ψ is the highest short root of R. Thus in $\widetilde{\mathcal{B}}'$, the counterparts of T_0 and U_j are $T_0' = Y^{\psi'^\vee} T(s_\psi)^{-1}$ and $U_k' = Y^{\pi_k} T(v(\pi_k))^{-1}$ for $k \in J'$.

The images of T_0' and U_j' under ω are $T(s_\psi)^{-1} X^{-\psi'^\vee}$ and $T(v(\pi_k))^{-1} X^{-\pi_k}$.

The duality theorem for double affine braid groups leads to the duality theorem for double affine Hecke algebras. This allows for the existence of intertwining operators, used to create polynomials that are Y^λ eigenvectors.

References

[1] I.G. Macdonald. *Affine Hecke algebras and orthogonal polynomials* Cambridge Tracts in Mathematics, vol 157, 2003.