LITTLEWOOD-RICHARDSON RULE AND ALCOVE WALKS
UNIVERSITY OF MELBOURNE PURE MATHS STUDENT SEMINAR
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MARTHA YIP
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WI 53706

0.1. **Introduction.** The symmetric group on n symbols S_n has presentation by generators $s_1, \dots, s_{n-1}$ and relations $s_i^2 = 1$, $(s_i s_{i+1})^3 = 1$.

S_n acts on the space of polynomials $\mathbb{C}[x_1, \dots, x_n]$ by switching the i and $i+1$ variables

$$s_i \cdot f(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i+1}, x_i, \dots, x_n).$$

The submodule of symmetric polynomials is

$$\mathbb{C}[x_1, \dots, x_n]^{S_n} = \{f \mid s_i f = f \text{ for all } 1 \le i \le n-1\}.$$

A basis for the symmetric polynomials is the basis of monomial symmetric functions

$$\{m_\lambda \mid \lambda \text{ a partition with at most } n \text{ parts}\},$$

where

$$m_\lambda = \sum_\alpha \mathbf{x}^\alpha$$

is the sum over all distinct permutations α of $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n)$. For example,

$$\begin{aligned}
m_{\square} &= x_1 + x_2 + x_3 \\
m_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} &= x_1x_2 + x_1x_3 + x_2x_3 \\
m_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} &= x_1x_2x_3 \\
m_{\square\square} &= x_1^2 + x_2^2 + x_3^2 \\
m_{\square\square\square} &= x_1^3 + x_2^3 + x_3^3 \\
m_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.
\end{aligned}$$

Another basis for the symmetric polynomials is the basis of (GL) Schur polynomials

$$\{s_\lambda \mid \lambda \text{ a partition with at most } n \text{ parts}\}.$$

For example,

$$\begin{aligned}
s_{\ydiagram{1}} &= x_1 + x_2 + x_3 \\
s_{\ydiagram{1,1}} &= x_1x_2 + x_2x_3 + x_1x_3 \\
s_{\ydiagram{1,1,1}} &= x_1x_2x_3 \\
s_{\ydiagram{2}} &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 \\
s_{\ydiagram{3}} &= x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + x_1x_2x_3 \\
s_{\ydiagram{2,1}} &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3.
\end{aligned}$$

There are other bases of symmetric polynomials that are easier to describe. Schur polynomials seem mysterious. Why do we study them?

0.2. **Schur polynomials are characters of GL_k representations.** Let V be a k-dimensional complex vector space. The group of automorphisms of V is the group of k by k invertible matrices $\mathrm{GL}(V) \cong \mathrm{GL}_k\mathbb{C}$.

A k-dimensional representation of a group G is a group homomorphism from G to k by k invertible matrices

$$\rho : G \to \mathrm{GL}(V) : g \mapsto \rho(g).$$

In other words, V is a G-module, with action

$$g \cdot x = \rho(g)x.$$

The character of the representation is a 1-dimensional representation

$$\chi : G \to \mathbb{C}^\times : g \mapsto \operatorname{tr} \rho(g).$$

A representation is irreducible if the only submodules of V are 0 and itself. Every representation is a direct sum of irreducible representations. Over $\mathbb{C}$, representations are isomorphic if and only if they have the same character. Characters encode useful information about representations.

The highest weight modules of GL_k are finite dimensional irreducible representations of GL_k. They are indexed by partitions with at most k parts. These can be realized as submodules of $V^{\otimes n}$, and their characters are the Schur functions.

Aside: Given $g \in GL(V)$, it has an action on $V^{\otimes n}$ by $g \cdot (v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$, so $V^{\otimes n}$ is a GL_k (and hence a SL_k)-module. Let L_λ denote the highest weight module of GL_k indexed by λ. The claim is that if the action of g on V has eigenvalues $x_1, \ldots, x_k$, then the action of g on L_λ has trace $s_\lambda(x_1, \ldots, x_k)$.

0.3. **What is a Littlewood-Richardson Rule?** We can tensor two highest weight GL_k modules together and get a new GL_k representation. Now, every representation is the direct sum of irreducible representations, so we might be interested in knowing how this direct sum decomposes.

$$L_\lambda \otimes L_\mu = \sum_\nu c_{\lambda\mu}^\nu L_\nu.$$

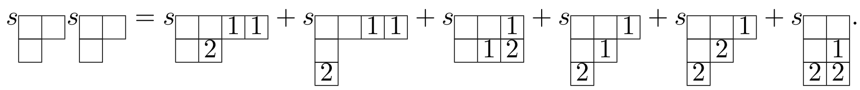
A Littlewood-Richardson rule tells us how to compute the coefficients $c_{\lambda\mu}^\nu$.

The same coefficients appear when we multiply the characters of these representations

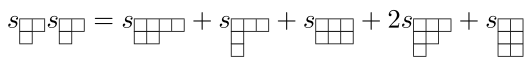
$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu.$$

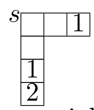
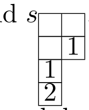
The original form of the Littlewood-Richardson rule was a combinatorial formula given in terms of fillings of tableaux.

A GL_3 example. If $\lambda = \mu =$ , then the partitions ν that appear in the sum are the ones obtainable by augmenting $\lambda =$  by a filling of type $\mu = (2, 1)$ so the rows are weakly increasing, the columns are strictly increasing, and the sequence obtained by reading the filling from right to left and top to bottom is a sequence so that for every initial segment, the number of is that appear is at least the number of $i+1$s that appear.



That is,



Note: There are two more admissable fillings,  and , but these give partitions with 4 parts. These do not correspond to nonzero highest weight modules for GL_3.

There is another combinatorial formula for the coefficients $c^{\nu}_{\lambda\mu}$.

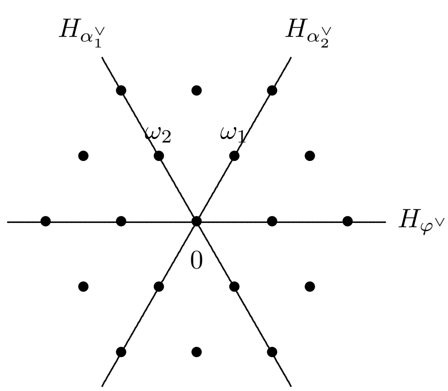
0.4. $c^{\nu}_{\lambda\mu}$ **in terms of paths.** Note: SL_k is the subgroup of GL_k of matrices with determinant one. All finite dimensional representations of SL_k are the highest weight modules L_λ. SL_k and GL_k characters for these highest weight modules are essentially the same.

SL_k is a compact semisimple Lie group (ie Type A). We can do representation theory for other compact semisimple Lie groups, and ask how tensor products of simple representations decompose into irreducible representations.

In 1994, Littelmann answered this (for the more general setting of complex symmetrizable Kac-Moody algebras) with the Littelmann path model.

The SL_3 example again. Let $V = \mathrm{span}_{\mathbb{R}}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, and let $\mathfrak{h}^*$ be the two dimensional subspace $\mathrm{span}_{\mathbb{R}}\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$. Let $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$.

S_3 acts on a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (the weight lattice) by reflections in hyperplanes

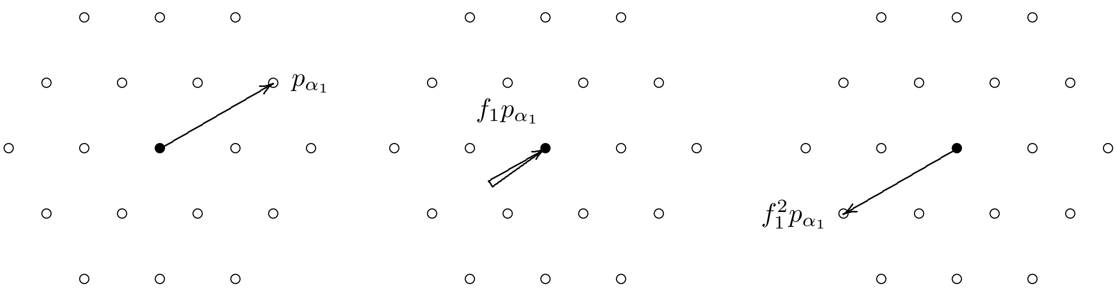


The dominant weights are the lattice points in the dominant cone C. The bijection between partitions with at most two parts and dominant weights is given by

$$(\lambda_1, \lambda_2) \leftrightarrow (\lambda_1 - \lambda_2)\omega_1 + \lambda_2\omega_2.$$

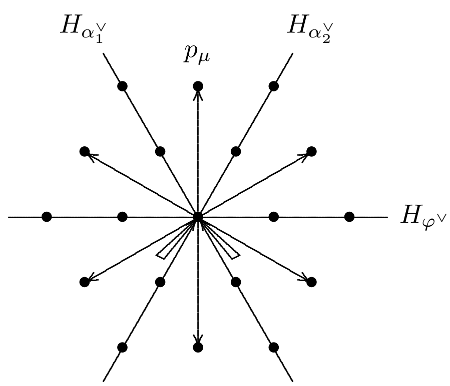
For example,  $\leftrightarrow \omega_1 + \omega_2$.

Let p_λ be a path from 0 to $\omega_1 + \omega_2$. The root operators f_1, f_2 and e_1, e_2 reflect and fold paths. For example, repeatedly applying f_1 to the path p_{α_1} gives



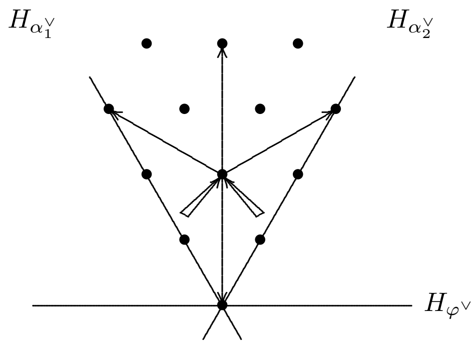
and $f_1^3 p_{\alpha_1} = 0$. The root operator e_1 reverses this.

A crystal is a set of paths that is closed under the action of root operators. For example, the crystal generated by p_μ is the following collection of eight paths



Then the weights that appear in the sum $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$ are the endpoints of the paths p_λ concatenated with all the paths in the crystal generated by p_μ that are contained in the dominant cone.

$$s_{\omega_1+\omega_2} s_{\omega_1+\omega_2} = s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_{\omega_1+\omega_2} + s_0.$$



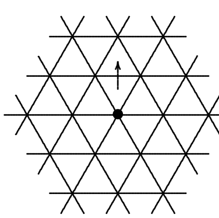
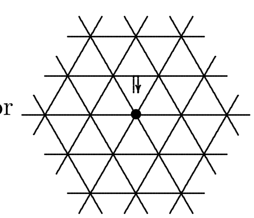
This matches up with the previous formula for GL_3 if you allow that $s_{\substack{\square\\\square\\\square}} = x_1x_2x_3 = 1$ in SL_3. Essentially that is the difference between their characters.

0.5. **Generalizations of paths and Schur functions.** Macdonald polynomials $P_\lambda(X, q, t)$ are symmetric functions with two parameters q, t. At $q = t$, the Type A Macdonald polynomials reduce to Schur polynomials.

For example,

$$P_{\omega_1+\omega_2} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + (2 + t + q + 2qt)\frac{1-t}{1-qt^2}.$$

The coefficients $c_{\lambda\mu}^\nu$ now depend on q and t. To find them, the Littelmann path model is generalized to the alcove walk model.

A step in an alcove walk is either a crossing from one alcove to an adjacent alcove, or a fold. A walk of type μ is a shortest walk from the fundamental alcove to the μ-hexagon, and we may fold it. So for example, a walk of type $\omega_1 + \omega_2$ is  or 

The sum

$$P_\lambda P_\mu = \sum_\nu c_{\lambda\mu}^\nu P_\nu$$

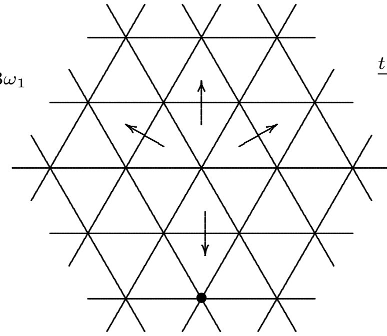
is taken over all walks of type μ with two-coloured folds, starting in any alcove of the λ-hexagon, that are contained in the dominant cone.

The coefficients $c_{\lambda\mu}^\nu$ depend on the folds of the walk, the beginning and ending alcoves of the walk, and any crossings that go backwards.

For example, the sum $P_{\omega_1+\omega_2}P_{\omega_1+\omega_2}$ is taken over 16 paths

$P_{2\omega_1+2\omega_2}$

$\frac{t^{1/2}-t^{-1/2}qt}{1-qt}P_{3\omega_1}$



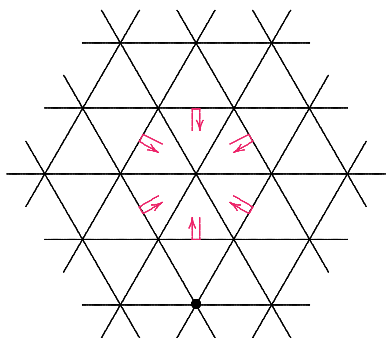
$\frac{t^{1/2}-t^{-1/2}qt}{1-qt}P_{3\omega_2}$

$\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\,\frac{t^{1/2}-t^{-1/2}q^2t^2}{1-q^2t^2}\,\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\,\frac{1-qt}{1-qt^2}\,\frac{1-qt^3}{1-qt^2}P_0$

$\frac{(t^{1/2}-t^{-1/2})q^3t^2}{1-q^3t^2}\,\frac{t^{-1/2}-t^{1/2}q^2t^2}{1-q^2t^2}\,\frac{t^{-1/2}-t^{1/2}qt}{1-qt}\,\frac{t^{-1/2}-t^{1/2}qt}{1-qt}P_{\omega_1+\omega_2}$

$2\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\,\frac{(t^{1/2}-t^{-1/2})q^2t}{1-q^2t}$

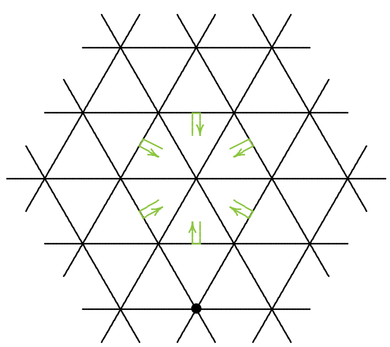
$\frac{t^{-1/2}-t^{1/2}qt}{1-qt}\,\frac{t^{-1/2}-t^{1/2}q^2t^2}{1-q^2t^2}P_{\omega_1+\omega_2}$



$2\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\,\frac{t^{1/2}-t^{-1/2}q^2t^2}{1-q^2t^2}$

$\frac{t^{-1/2}-t^{1/2}}{1-t}\,\frac{t^{-1/2}-t^{1/2}qt}{1-qt}P_{\omega_1+\omega_2}$

$\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\,\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\,\frac{t^{1/2}-t^{-1/2}q^2t^2}{1-q^2t^2}\,\frac{t^{-1/2}-t^{1/2}}{1-qt^2}P_{\omega_1+\omega_2}$



$\frac{(t^{-1/2}-t^{1/2})qt^2}{1-qt^2}(t^{-3/2}+2t^{-1/2}+2t^{1/2}+t^{3/2})P_{\omega_1+\omega_2}$

At $q=0$, this reduces to the Littlewood-Richardson rule for Hall-Littlewood polynomials

$$P_{\omega_1+\omega_2}P_{\omega_1+\omega_2} = P_{2\omega_1+2\omega_2} + t^{1/2}P_{3\omega_1} + t^{1/2}P_{3\omega_2} + (2+t(1-t))P_{\omega_1+\omega_2} + t^{3/2}P_0.$$

Furthermore, since the Schur polynomial is given by $s_\mu = t^{\ell(w_\mu)/2}P_\mu(0,t)|_{t=0}$, where w_μ is the longest element in the stabilizer of μ in S_3, then at $q=t=0$, this reduces to the Littlewood-Richardson rule for Schur polynomials

$$s_{\omega_1+\omega_2}s_{\omega_1+\omega_2} = s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_{\omega_1+\omega_2} + s_0.$$

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