Macdonald polynomials and alcove walks

Martha Yip

June 6 2009

Schur polynomials form a basis for the space of symmetric functions.

$$
\begin{aligned}
s_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}} &= \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \\
&= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3.
\end{aligned}
$$

The classical Littlewood-Richardson rule is a combinatorial formula for calculating the structure coefficients in the Schur basis.

$$s_\mu s_\lambda = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu.$$

The LR rule is given in terms of fillings of Young diagrams.

$$s_{\ytableau{\ & \ & 1 & 1 \\ \ & 2}},\quad s_{\ytableau{\ & \ & 1 & 1 \\ \ \\ 2}},\quad s_{\ytableau{\ & \ & 1 \\ \ & 1 & 2}},\quad s_{\ytableau{\ & \ & 1 \\ \ & 1 \\ 2}},\quad s_{\ytableau{\ & \ & 1 \\ \ & 2 \\ 2}},\quad s_{\ytableau{\ & \ \\ \ & 1 \\ 2 & 2}}$$

$$s_{\ydiagram{2,1}}s_{\ydiagram{2,1}} = s_{\ydiagram{4,2}} + s_{\ydiagram{4,1,1}} + s_{\ydiagram{3,3}} + 2s_{\ydiagram{3,2,1}} + s_{\ydiagram{2,2,2}}.$$

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The Littelmann path model provides a generalization to complex symmetrizable Kac-Moody algebras.

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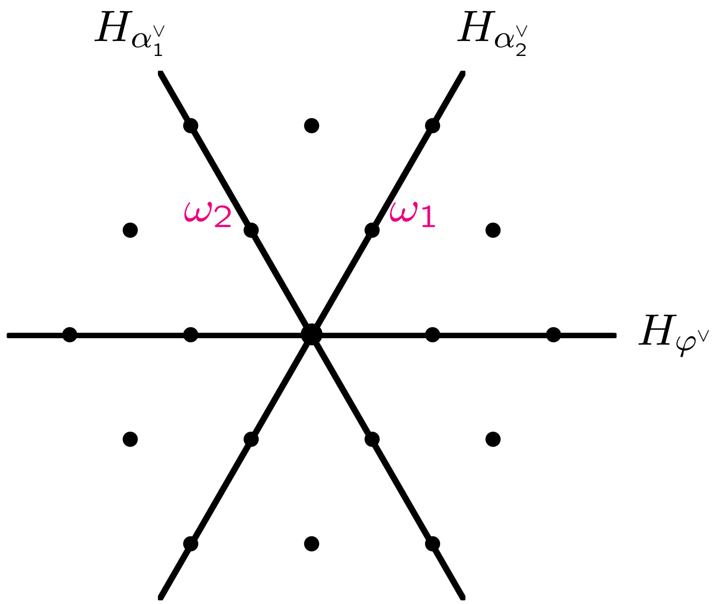
An SL_3 example

Let $V = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3\}/\mathrm{span}_{\mathbb{R}}\{e_1 + e_2 + e_3\}$.
Let $L = \mathrm{span}_{\mathbb{Z}}\{\omega_1, \omega_2\} \subseteq V^*$.

The Weyl group $S_3 \subseteq \mathrm{GL}(V^*)$ is a finite reflection group that acts on L.



Dictionary

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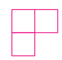	$\leftrightarrow$	$\omega_1 + \omega_2$

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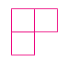	$\leftrightarrow$	$\omega_1 + \omega_2$
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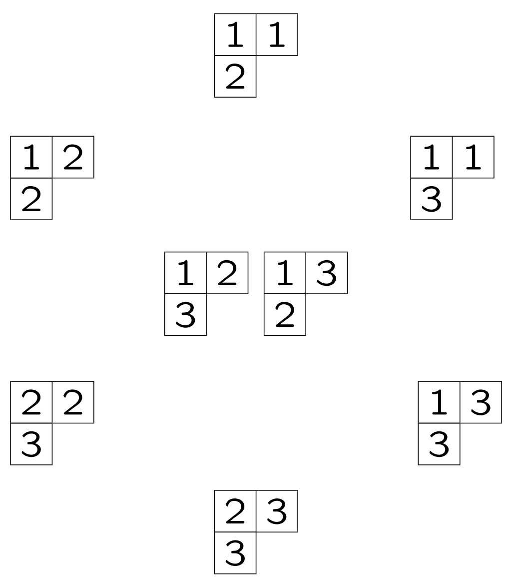
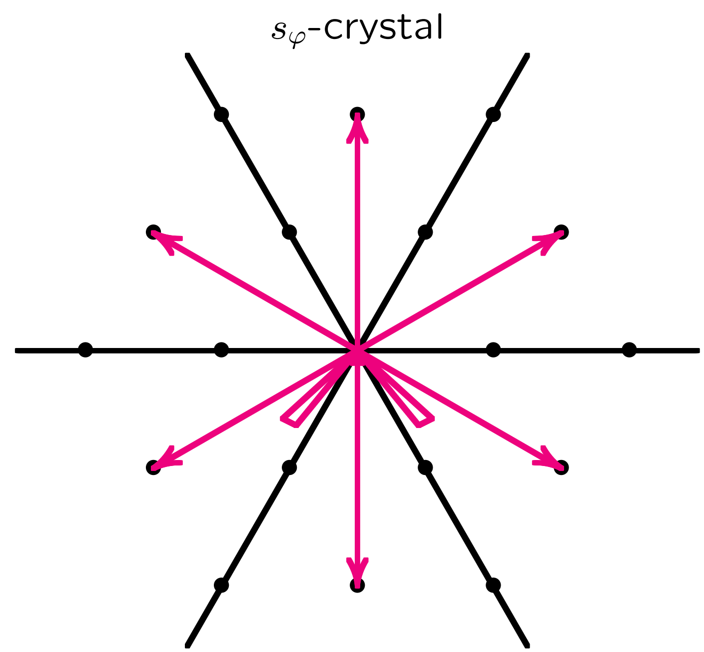
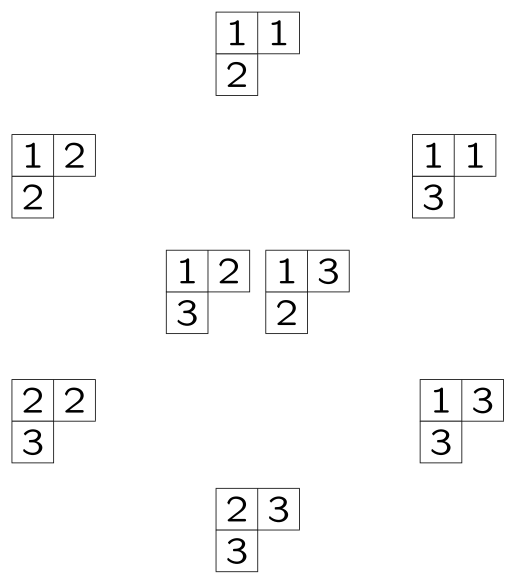


$\longleftrightarrow$



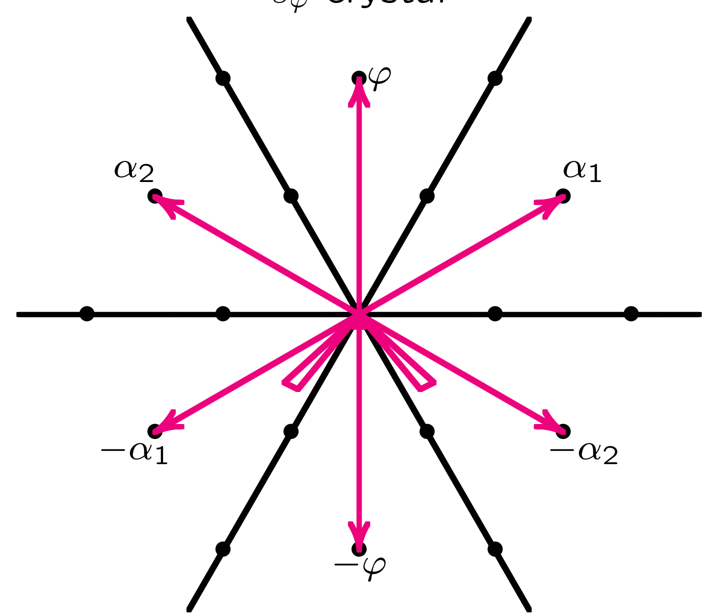
$$s_{\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \longleftrightarrow s_\varphi = x^{\varphi} + x^{\alpha_2} + x^{\alpha_1} + x^{-\alpha_2}$$

$$+ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad + x^{-\varphi} + x^{-\alpha_1} + x^{0} + x^{0}$$

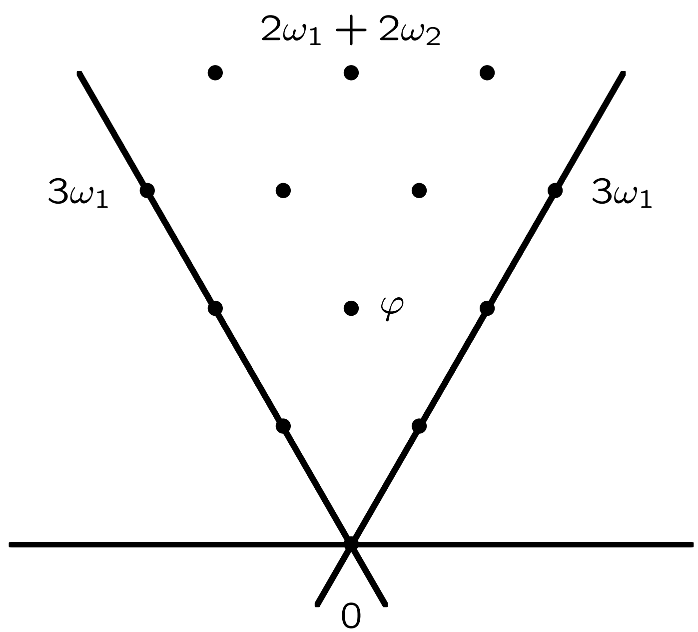


$\longleftrightarrow$

s_φ-crystal



$s_\mu s_\lambda = \sum_p s_{\mathrm{end}(p)}$ is the sum over paths p in the crystal of s_μ, starting at λ, contained in the dominant cone.



$$
\begin{aligned}
s_\varphi s_\varphi &= s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_\varphi + s_0 \\
s_{(2,1)} s_{(2,1)} &= s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(2,2,2)}.
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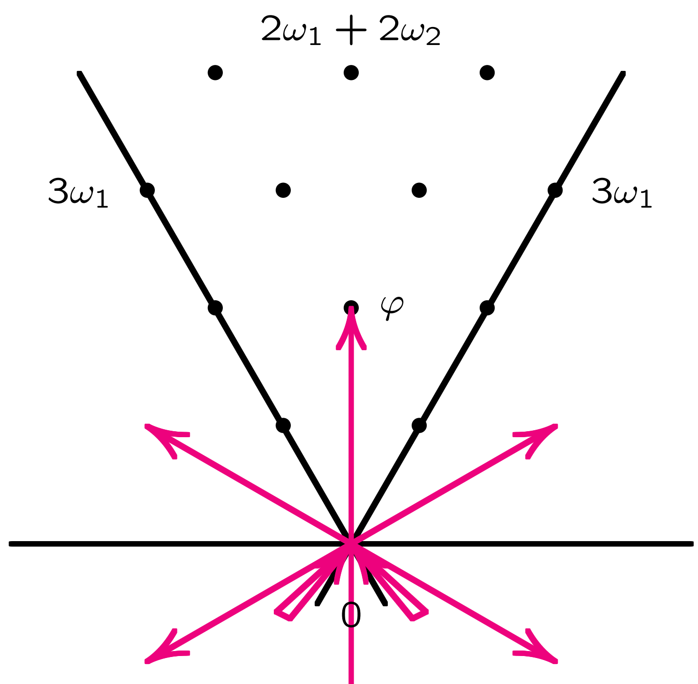

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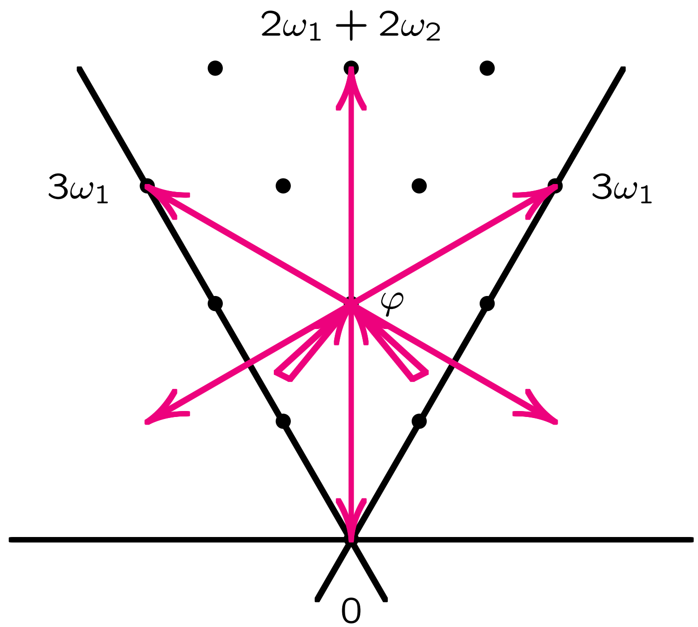

$$s_\varphi s_\varphi \quad = \quad s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_\varphi + s_0$$
$$s_{\yng(2,1)} s_{\yng(2,1)} \quad = \quad s_{\yng(4,2)} + s_{\yng(4,1,1)} + s_{\yng(3,3)} + 2s_{\yng(3,2,1)} + s_{\yng(2,2,2)}.$$

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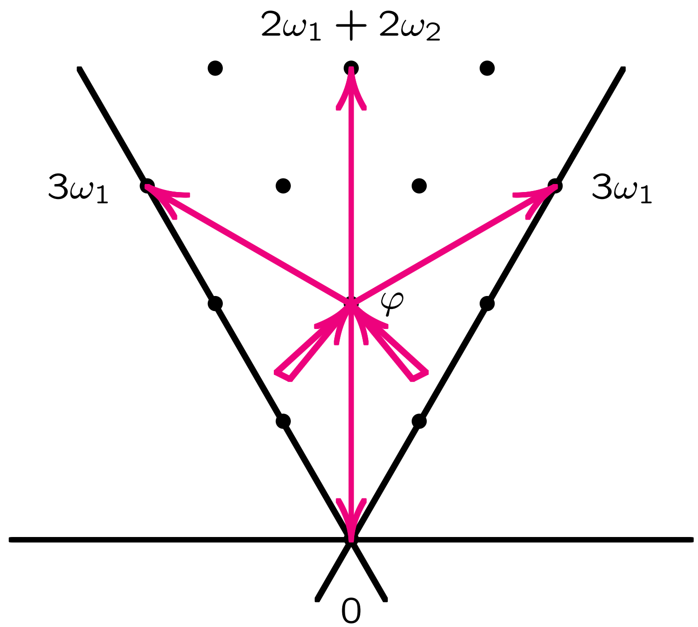

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Symmetric Polynomials

$P_\lambda(q,t)$ are the Macdonald polynomials.

$P_\lambda(0,0)$ are the Schur polynomials.

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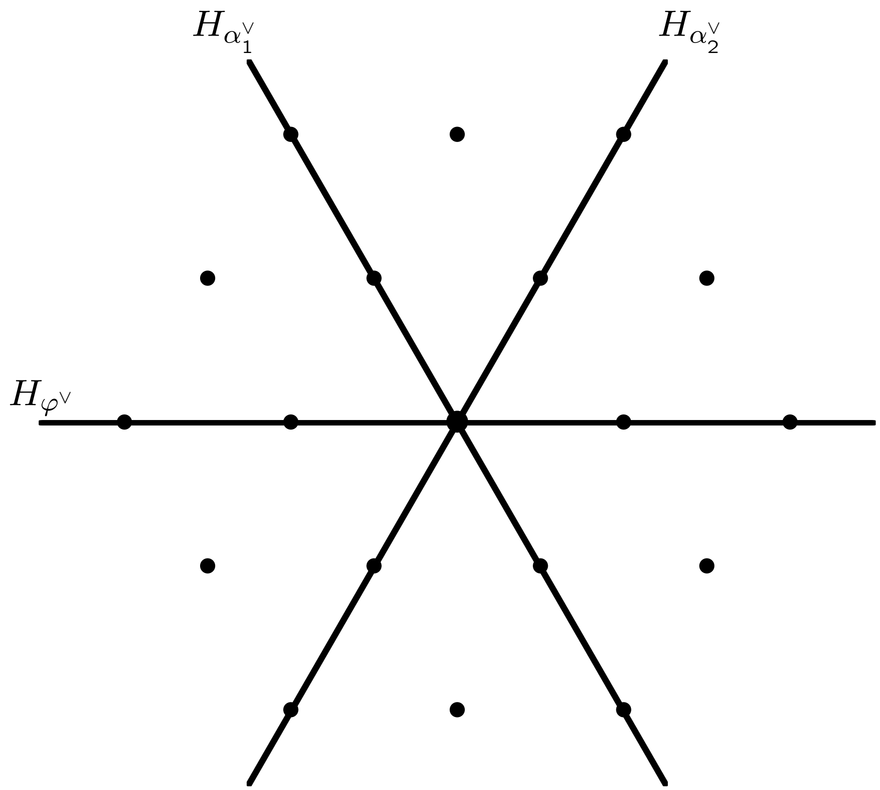
$$P_\varphi(q,t) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + (2 + t + q + 2qt)\frac{1-t}{1-qt^2}x^0$$

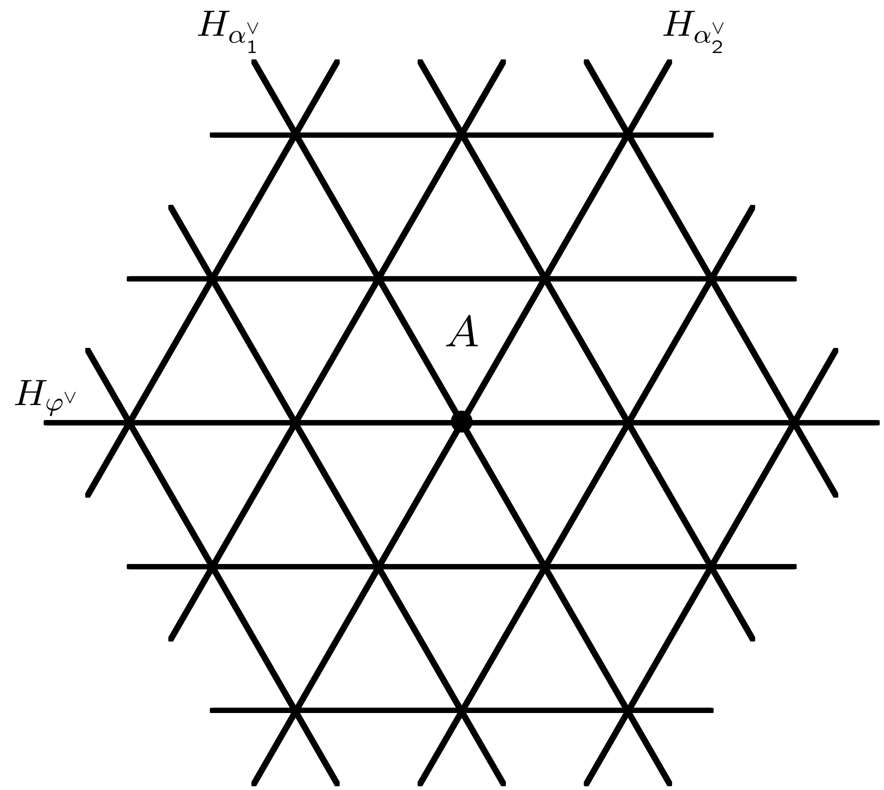
$P_\lambda(0,0)$ are the Schur polynomials.

$$P_\varphi(0,0) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + 2x^0$$

Alcove walks are the analogue of paths for Macdonald polynomials.

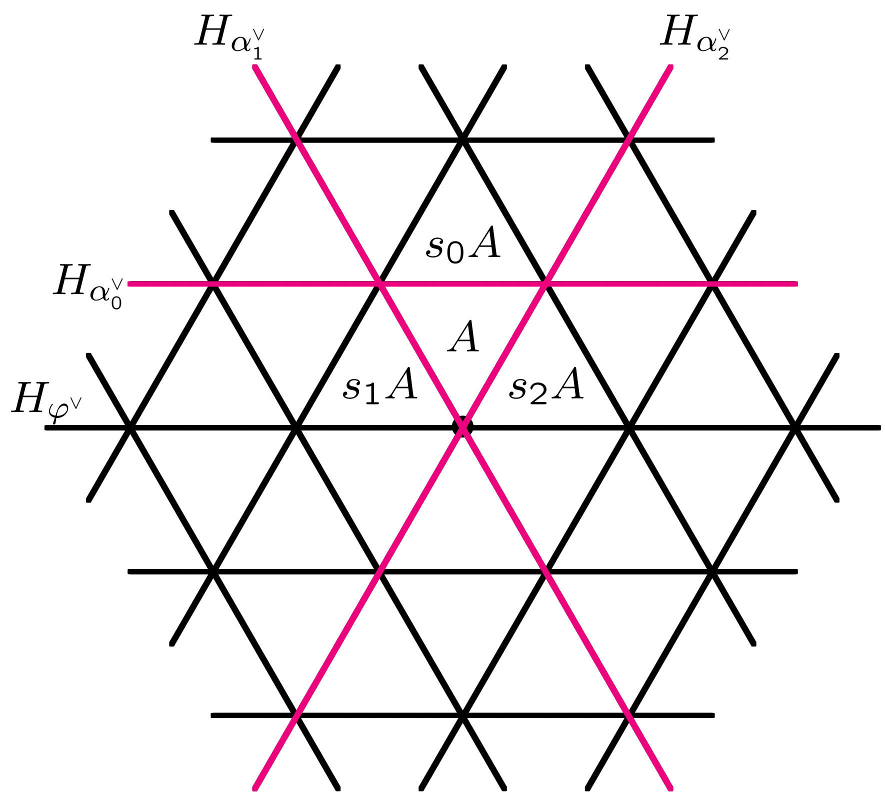


Weyl group S_3 acts on vector space V^* by reflections.



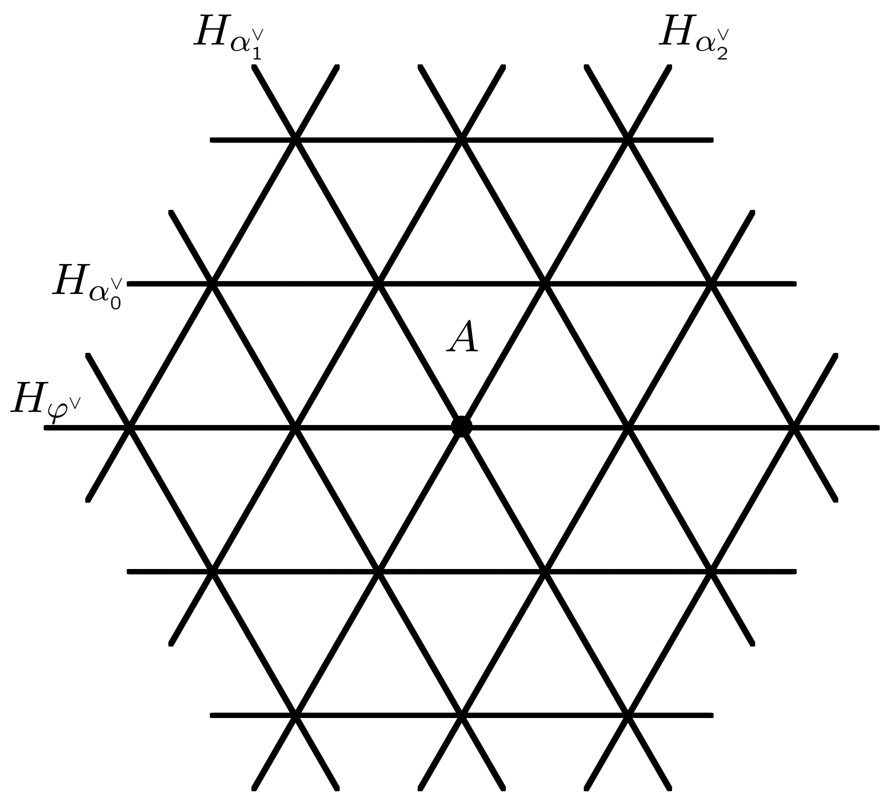
Affine Weyl group $W^\vee = X \rtimes S_3$ acts on vector space V^* by reflections and translations.

A is the fundamental alcove.



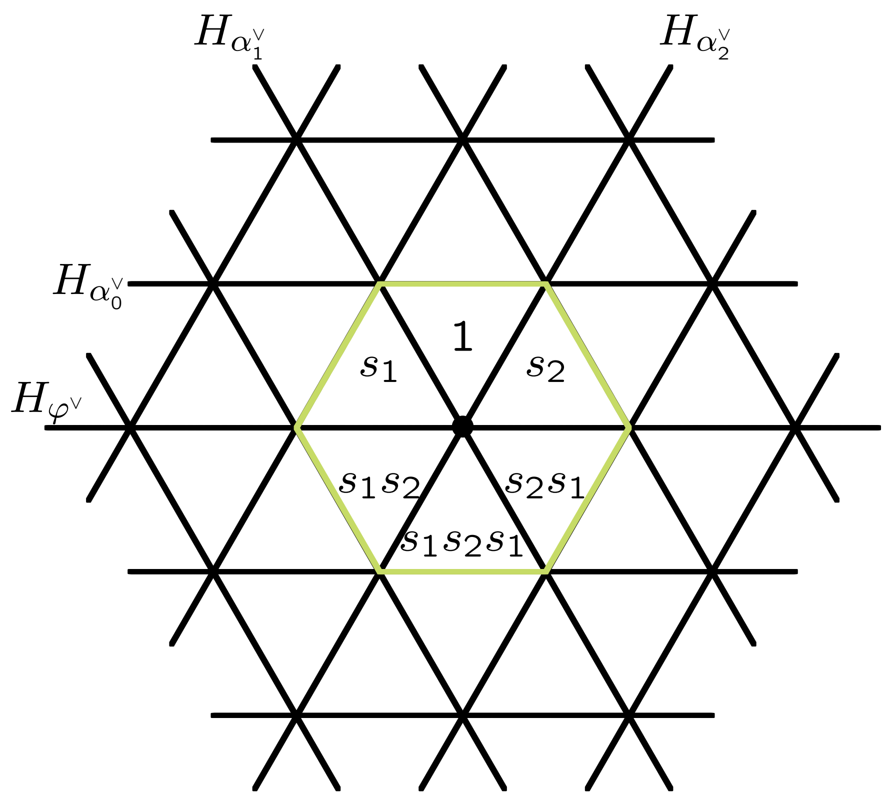
Walls of A are $H_{\alpha_0^\vee}, H_{\alpha_1^\vee}, H_{\alpha_2^\vee}$.

s_0, s_1, s_2 are reflections.

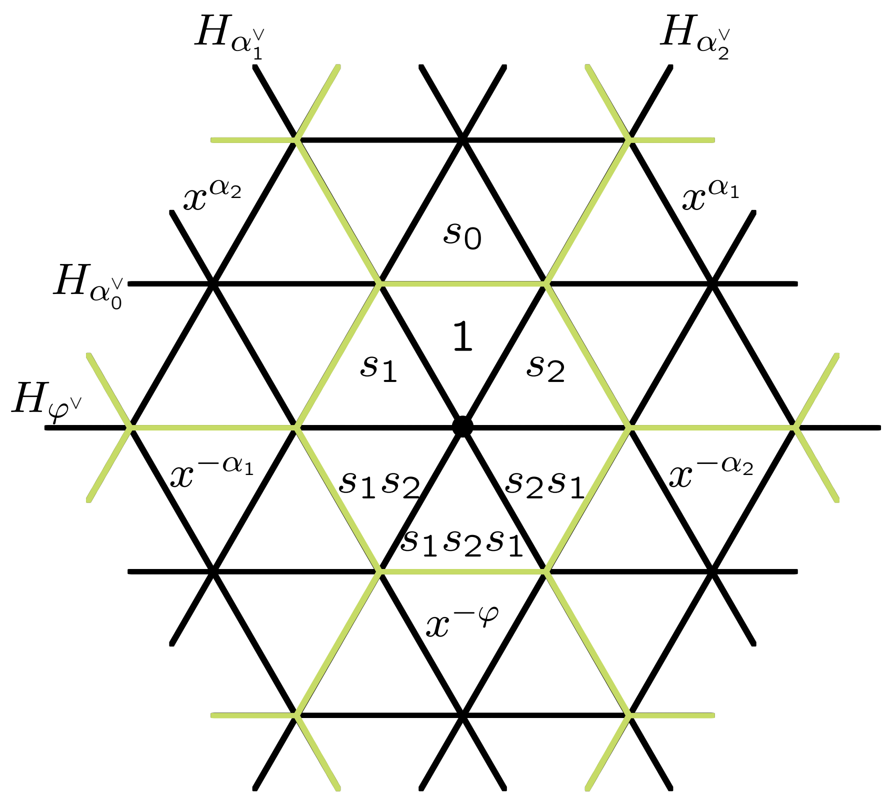


$$W^\vee \longleftrightarrow \text{alcoves}$$

$$w \leftrightarrow w^{-1}A$$

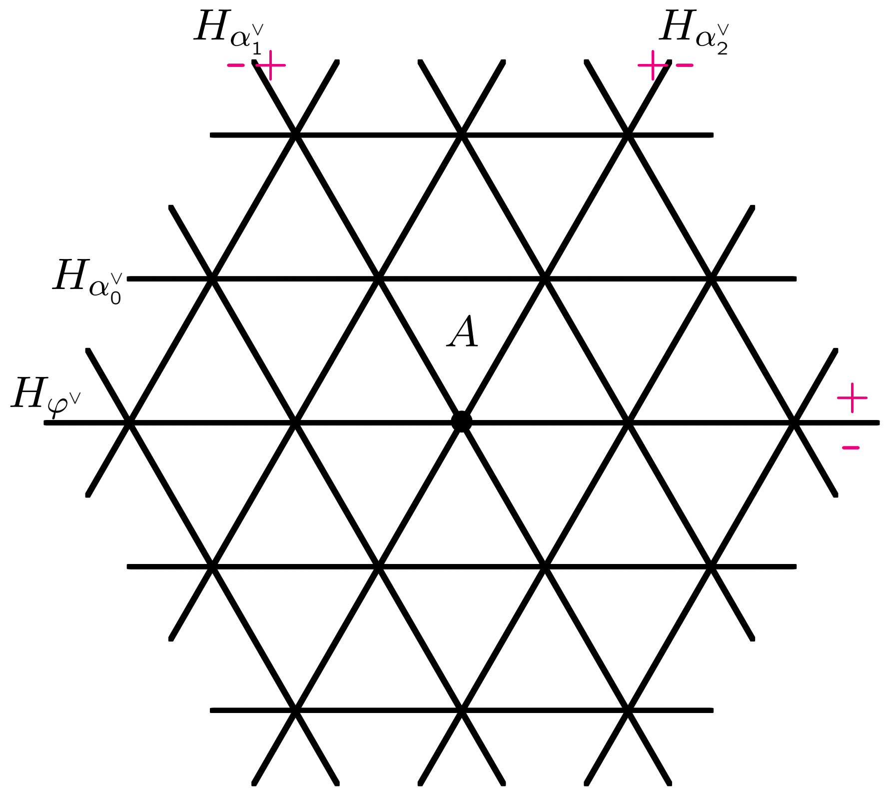


$S^3 \longleftrightarrow$ alcoves in 0-hexagon



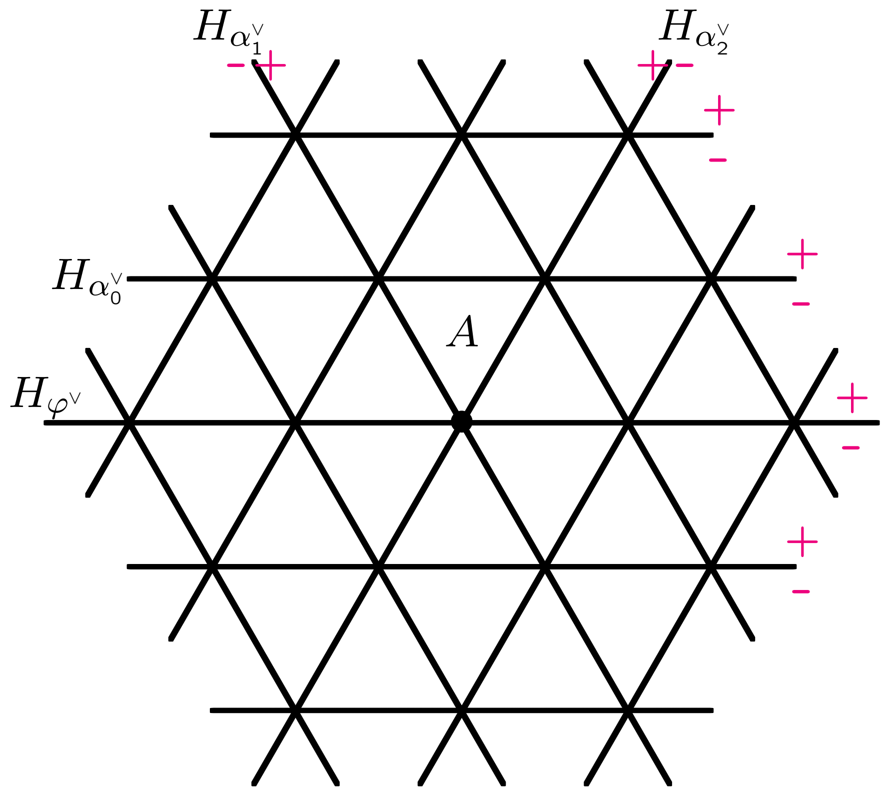
$S^3 \longleftrightarrow$ alcoves in 0-hexagon

$X \longleftrightarrow$ top alcoves in hexagons



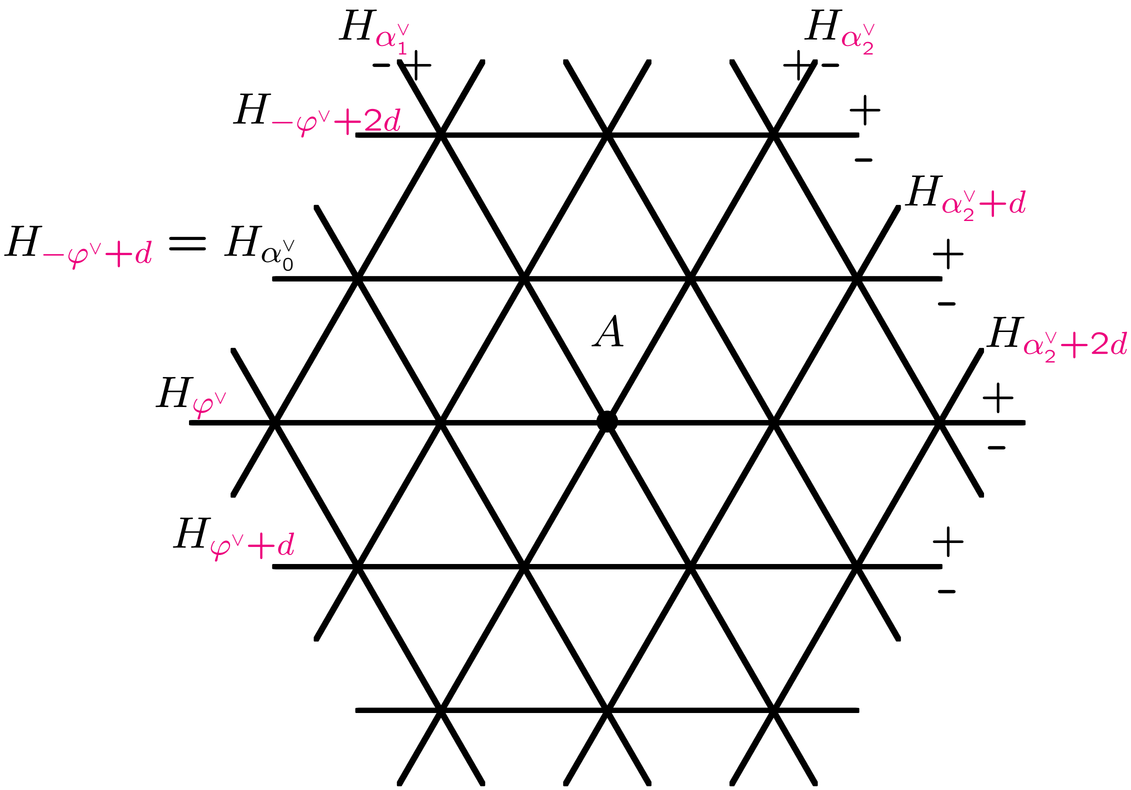
Periodic orientation on the walls:

1. A is on positive side of hyperplanes through 0,



Periodic orientation on the walls:

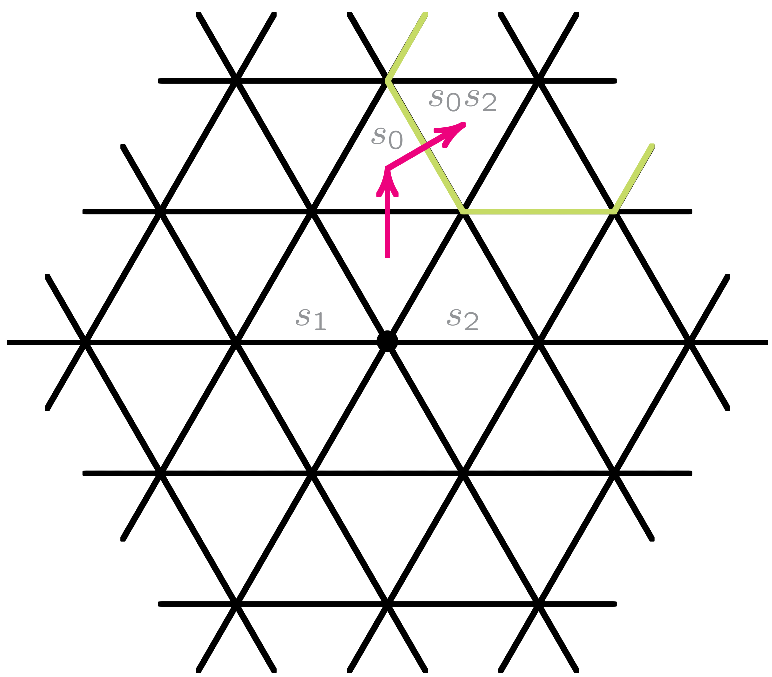
1. A is on positive side of hyperplanes through 0,
2. parallel hyperplanes have same orientation.



Hyperplane labels (positive affine coroots)

Walks and foldings

Let $\mu \in L$. Let $m_\mu = s_{i_1} \cdots s_{i_r}$ be a minimal length walk to the μ-hexagon.



$\mu = 2\omega_1$

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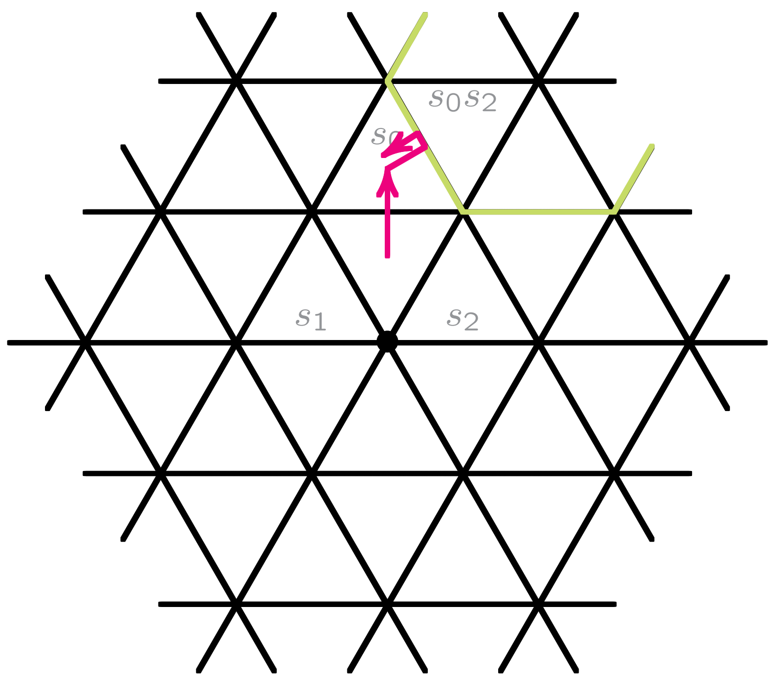

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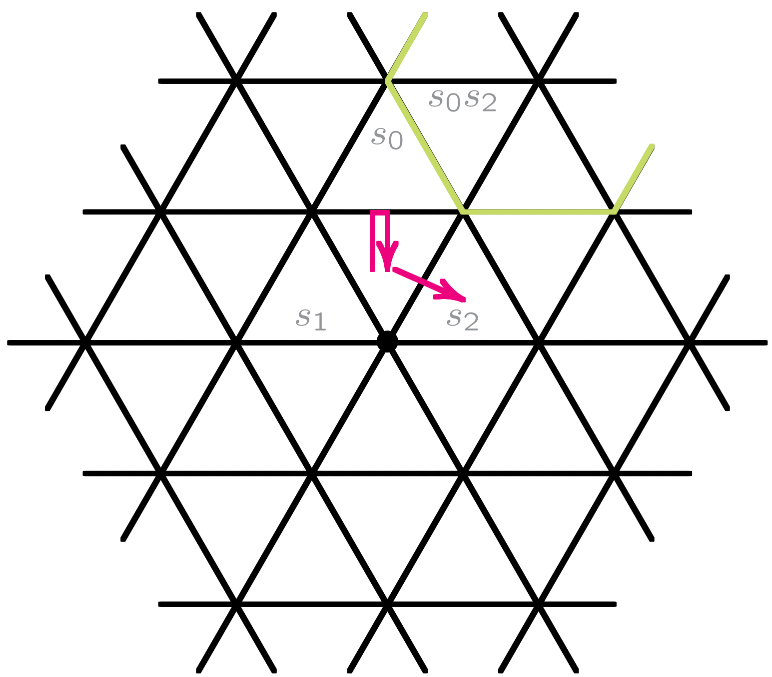

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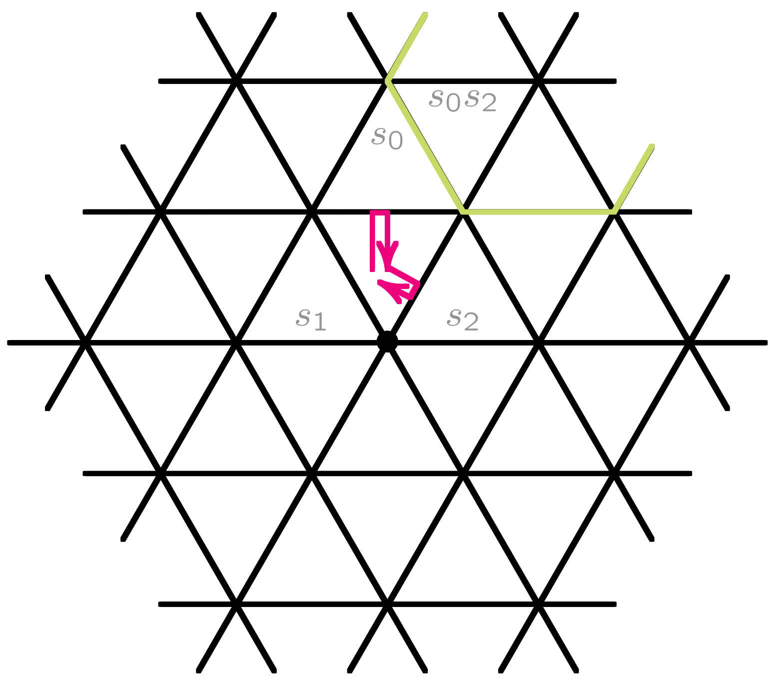

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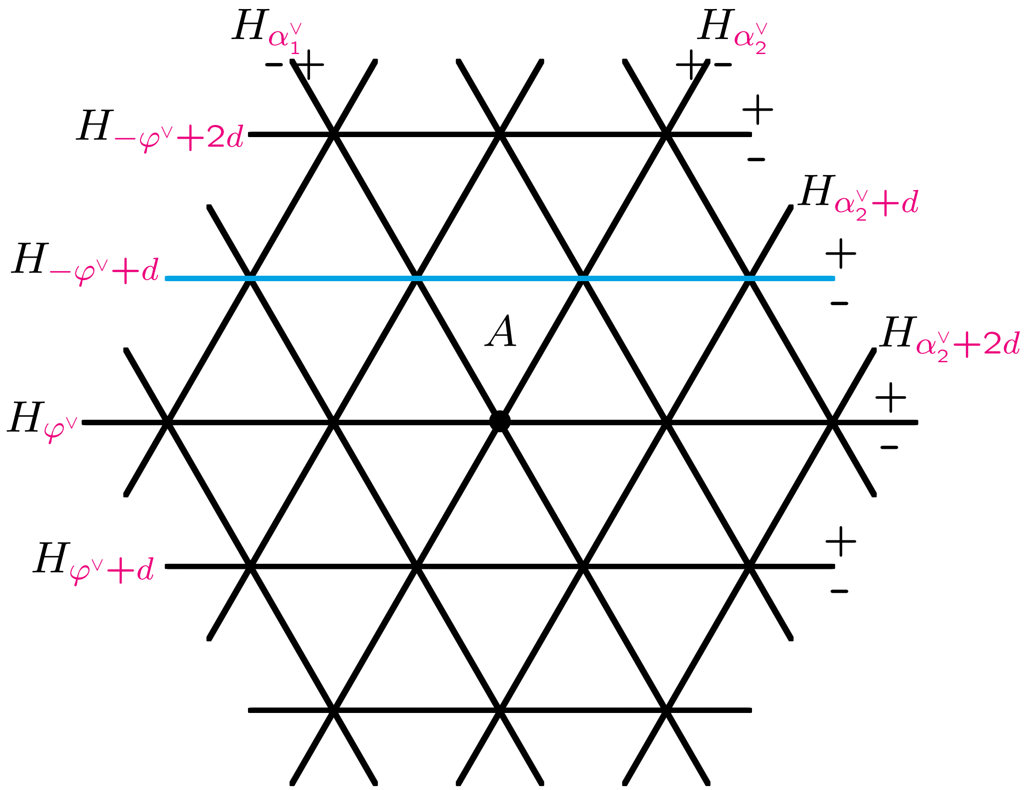
Theorem (Ram, Y) The Macdonald polynomial is

$$P_\mu(q,t) = \sum_{\substack{\text{foldings } p \text{ of } m_\mu \\ \mathsf{i}(p)\in S_3}} X^{\mathsf{wt}(p)} t^{\ell(\mathsf{d}(p)) - \ell(w_0 \mathsf{i}(p))/2} f_p,$$

where

$$f_p = \prod_{k\text{th step a fold}} \frac{t^{-1/2} - t^{1/2}}{1 - q^{\mathsf{sh}(b_k)} t^{\mathsf{ht}(b_k)}} \prod_{k\text{th step a - fold}} q^{\mathsf{sh}(b_k)} t^{\mathsf{ht}(b_k)},$$

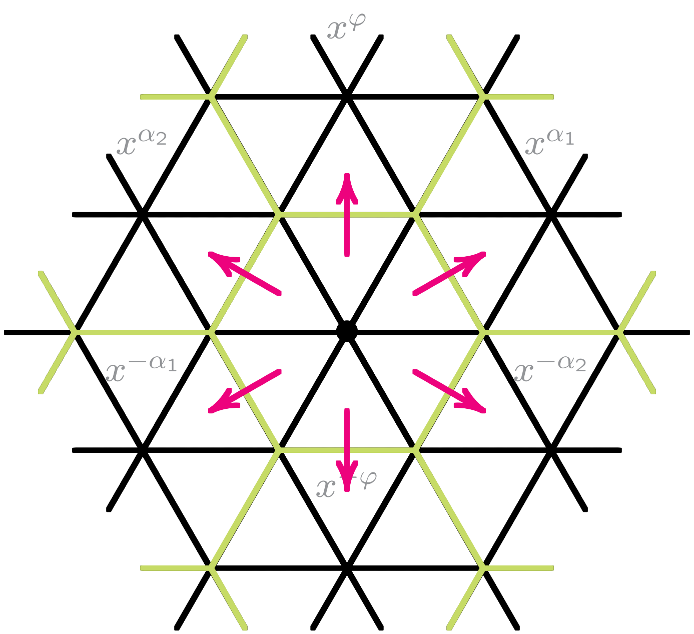
and $-b_k$ is the hyperplane crossed by the kth last step in the reverse walk m_μ^{-1}.



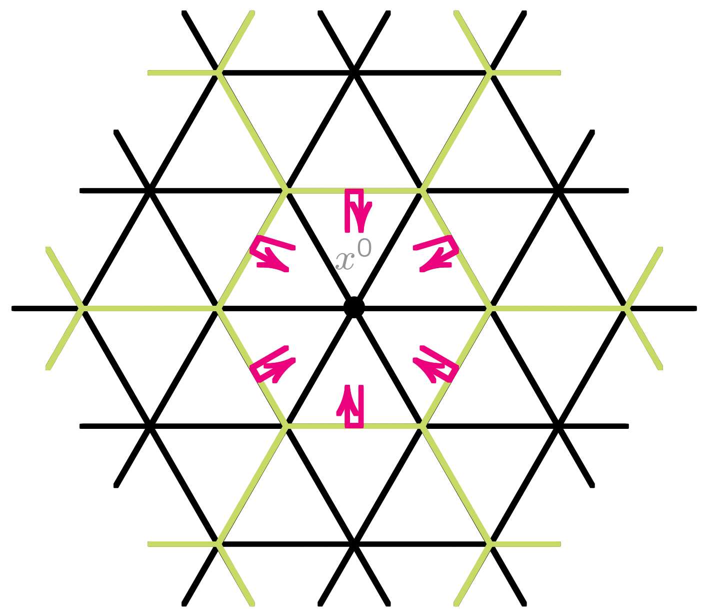
$$\mathrm{ht}(\varphi^\vee - d) = 2 \qquad \mathrm{sh}(\varphi^\vee - d) = 1.$$

Example

$$P_\varphi(q,t) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + (2+t+q+2qt)\frac{1-t}{1-qt^2}x^0$$



M_φ



$(2+t+q+2qt)\frac{1-t}{1-qt^2}$

Example

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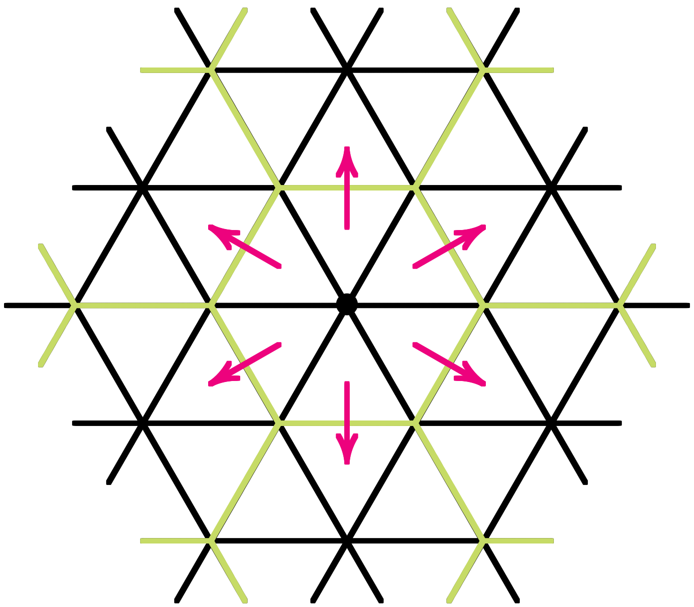
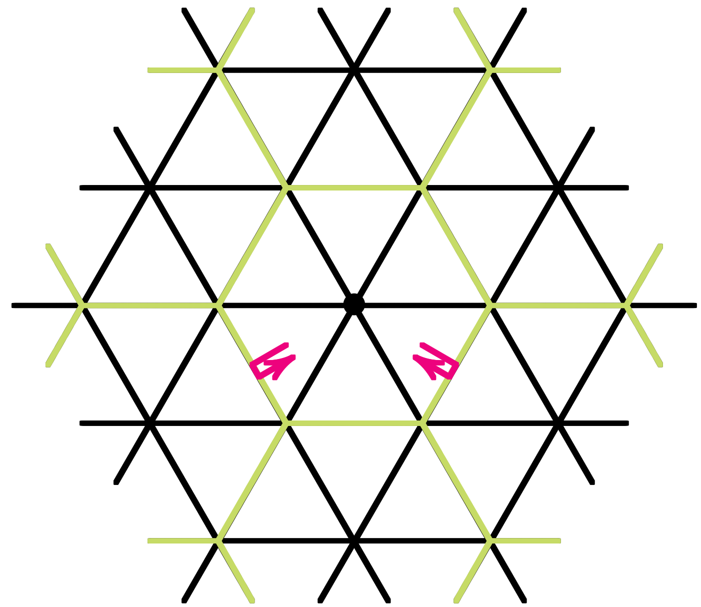
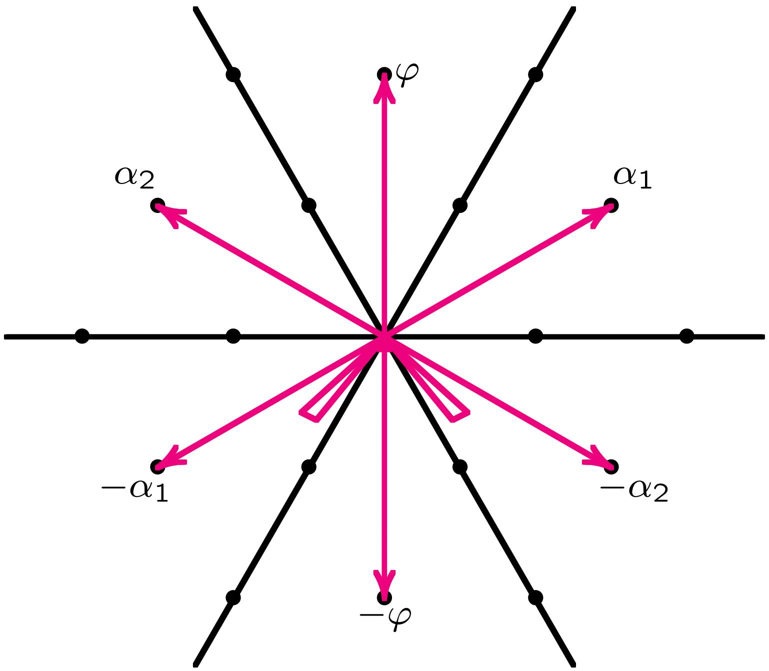


M_φ



$(2+0+0+0)\frac{1-0}{1-0}$

s_φ-crystal



Theorem (Y)

$$P_\mu(q,t)P_\lambda(q,t) = \sum_{\substack{p\in\mathcal{B}(m_\mu, wm_\lambda) \\ p\subseteq\overline{C}, w\in W_\lambda}} c_p P_{-w_0\mathrm{wt}(p)}(q,t)$$

is the sum over walks of type m_μ^{-1} starting in the λ-hexagon, contained in the dominant cone, whose folds are two-coloured, where

$$c_p = b_p e_p f_p f_p n_p,$$

$$b_p = \prod_{a^\vee \in m_\lambda^{-1}\mathcal{L}(\mathsf{i}(p))} \frac{t_{a^\vee}^{1/2} - t_{a^\vee}^{-1/2} q^{\mathsf{sh}(-a^\vee)} t^{\mathsf{ht}(-a^\vee)}}{1 - q^{\mathsf{sh}(-a^\vee)} t^{\mathsf{ht}(-a^\vee)}},$$

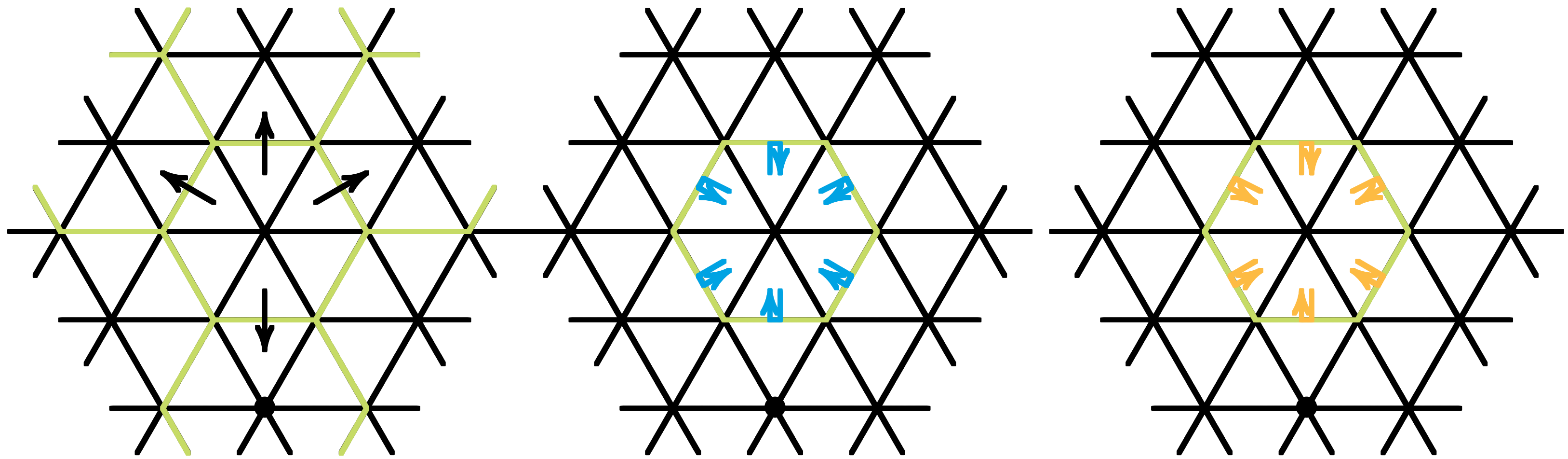
$$e_p = \prod_{a^\vee \in m_{-w_0 \mathsf{wt}(p)}^{-1}\mathcal{L}(\mathsf{c}(p)^{-1})} \frac{t_{a^\vee}^{-1/2} - t_{a^\vee}^{1/2} q^{\mathsf{sh}(-a^\vee)} t^{\mathsf{ht}(-a^\vee)}}{1 - q^{\mathsf{sh}(-a^\vee)} t^{\mathsf{ht}(-a^\vee)}},$$

$$f_p = \prod_{k \in \phi(p)} \frac{t_{b_k^\vee}^{1/2} - t_{b_k^\vee}^{-1/2}}{1 - q^{\mathsf{sh}(-b_k^\vee)} t^{\mathsf{ht}(-b_k^\vee)}} \prod_{k \in \phi^-(p)} q^{\mathsf{sh}(-b_k^\vee)} t^{\mathsf{ht}(-b_k^\vee)},$$

$$f_p = \prod_{k \in \phi(p)'} \frac{t_{c_k^\vee}^{1/2} - t_{c_k^\vee}^{-1/2}}{1 - q^{\mathsf{sh}(-c_k^\vee)} t^{\mathsf{ht}(-c_k^\vee)}} \prod_{k \in \phi(p)', r-k+1 \in \xi^+(p)} q^{\mathsf{sh}(-c_k^\vee)} t^{\mathsf{ht}(-c_k^\vee)},$$

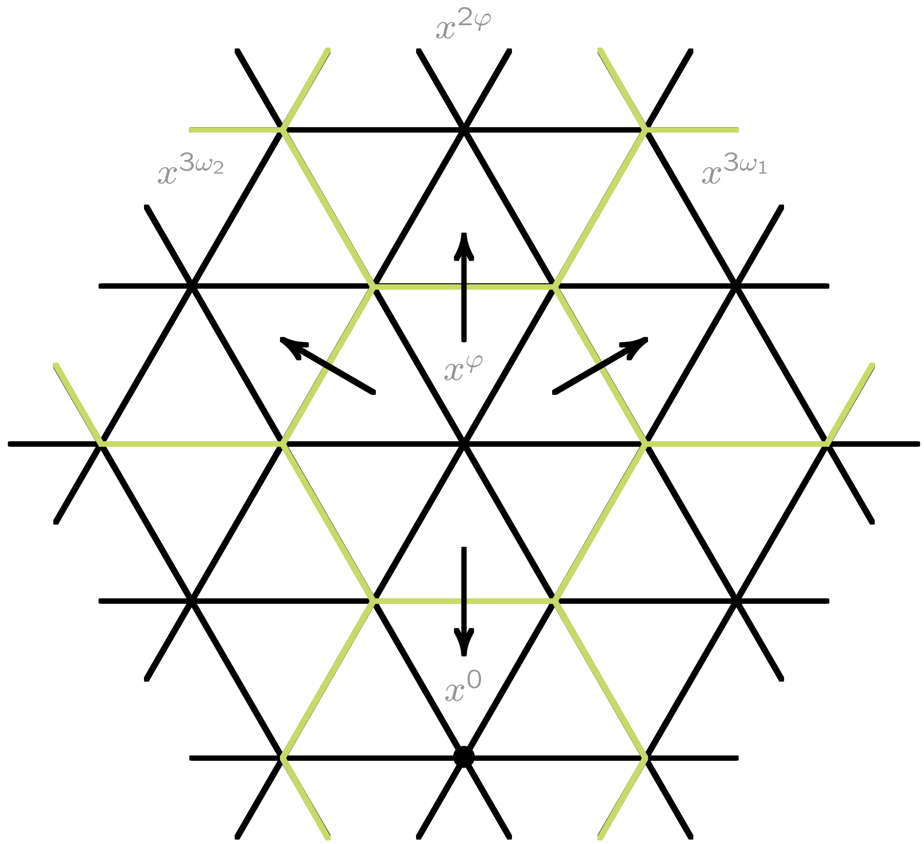
$$n_p = \prod_{j \in \xi^-(p)} \frac{1 - q^{\mathsf{sh}(-h_j^\vee)} t^{\mathsf{ht}(-h_j^\vee)-1}}{1 - q^{\mathsf{sh}(-h_j^\vee)} t^{\mathsf{ht}(-h_j^\vee)}} \frac{1 - q^{\mathsf{sh}(-h_j^\vee)} t^{\mathsf{ht}(-h_j^\vee)+1}}{1 - q^{\mathsf{sh}(-h_j^\vee)} t^{\mathsf{ht}(-h_j^\vee)}}.$$

Example

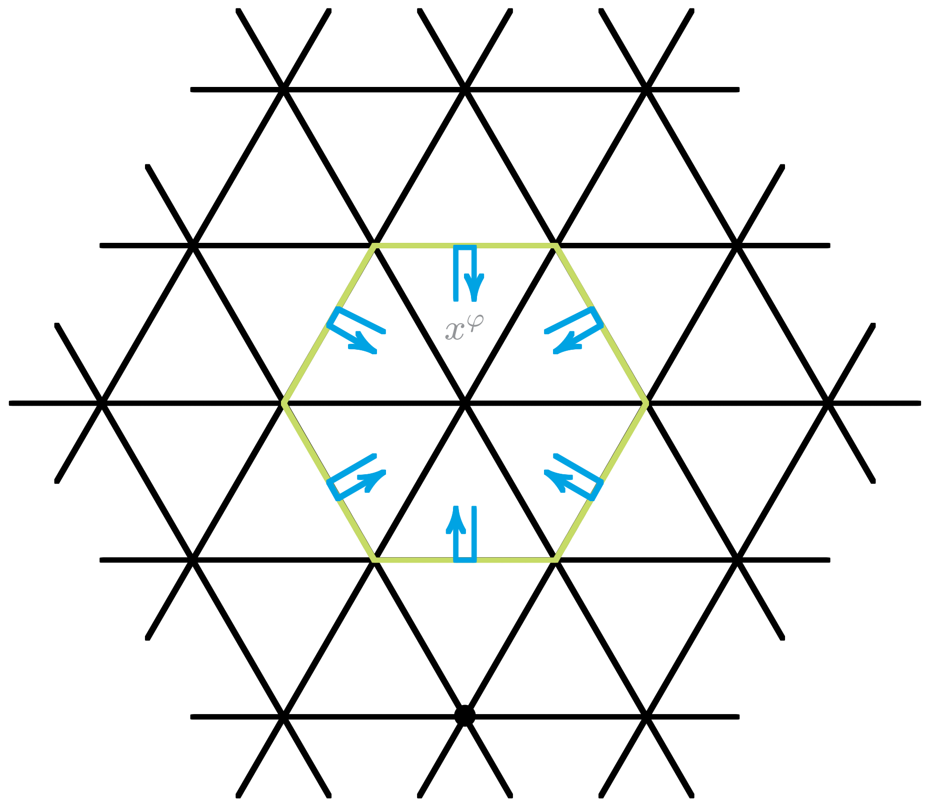


$$P_\varphi(q,t)P_\varphi(q,t)$$

$$P_\varphi(q,t)P_\varphi(q,t) =$$

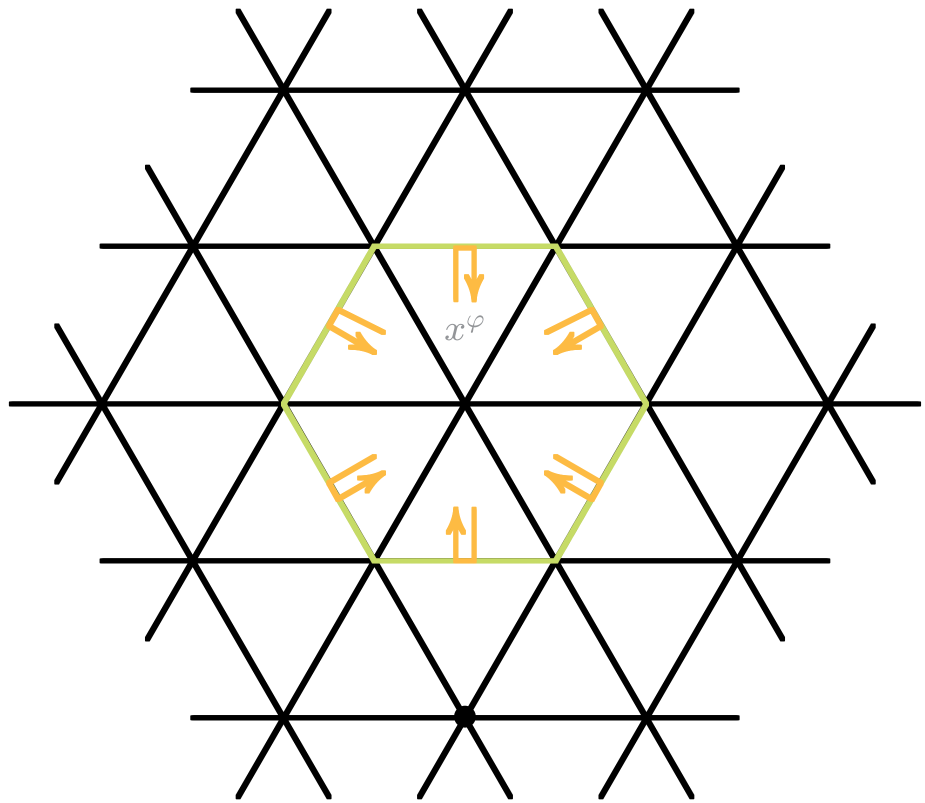


$$P_{2\varphi} + \frac{1-q}{1-qt}t^{1/2}P_{3\omega_1} + \frac{1-q}{1-qt}t^{1/2}P_{3\omega_2} + \frac{1-q}{1-qt}\frac{1-q^2t}{1-q^2t^2}\frac{1-q}{1-qt}\frac{1-qt}{1-qt^2}\frac{1-qt^3}{1-qt^2}t^{3/2}P_0 + \cdots$$



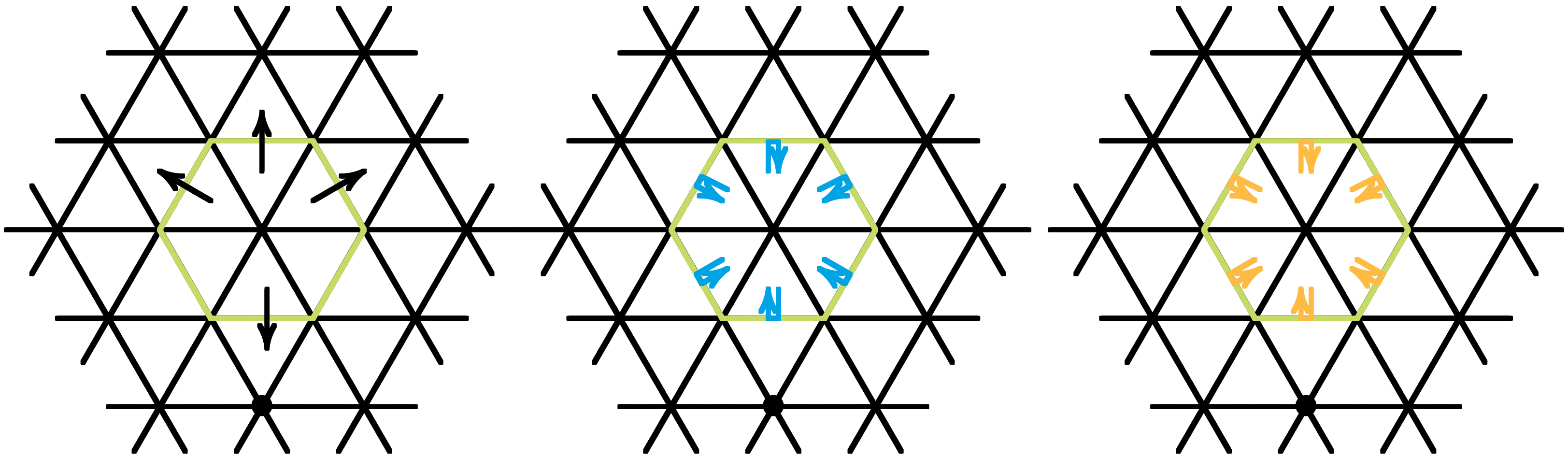
$$\frac{(t^{1/2}-t^{-1/2})q^3t^2}{1-q^3t^2}\frac{t^{-1/2}-t^{1/2}qt}{1-qt}\frac{t^{-1/2}-t^{1/2}q^2t^2}{1-q^2t^2}\frac{t^{-1/2}-t^{1/2}qt}{1-qt}P_\varphi + 2\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\frac{(t^{1/2}-t^{-1/2})q^2t}{1-q^2t}\frac{t^{-1/2}-t^{1/2}qt}{1-qt}\frac{t^{-1/2}-t^{1/2}q^2t^2}{1-q^2t^2}P_\varphi$$

$$2\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\frac{t^{1/2}-t^{-1/2}q^2t^2}{1-q^2t^2}\frac{t^{-1/2}-t^{1/2}}{1-t}\frac{t^{-1/2}-t^{1/2}qt}{1-qt}P_\varphi + \frac{t^{1/2}-t^{-1/2}qt}{1-qt}\frac{t^{1/2}-t^{-1/2}qt}{1-qt}\frac{t^{1/2}-t^{-1/2}q^2t^2}{1-q^2t^2}\frac{t^{-1/2}-t^{1/2}}{1-qt^2}P_\varphi + \cdots$$



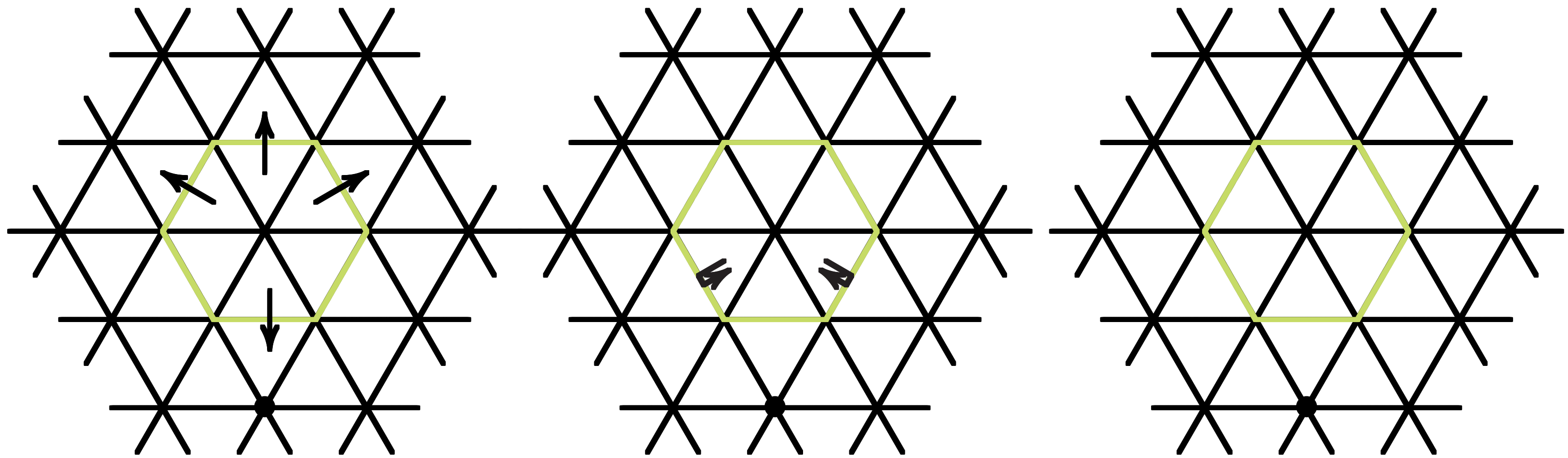
$$q\frac{1-t}{1-qt^2}(1+2t+2t^2+t^3)P_{\varphi}.$$

Bi-coloured alcove walks for Macdonald polynomials ...

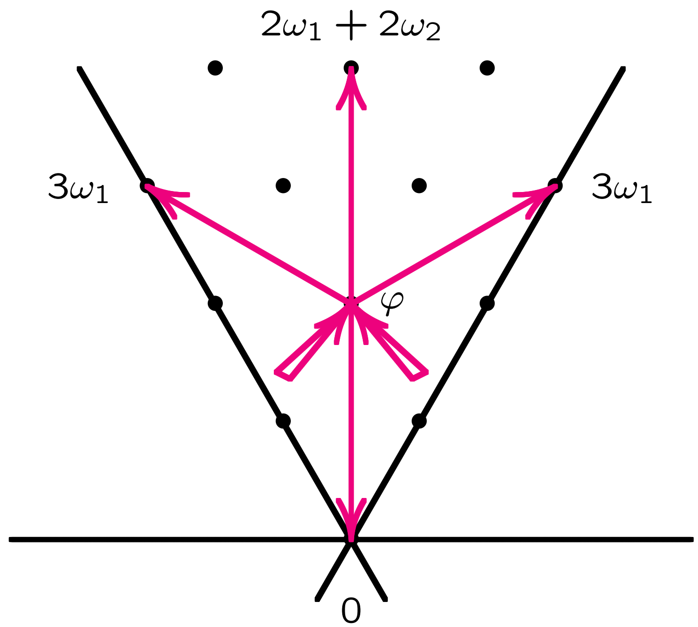


$$P_\varphi(q,t)P_\varphi(q,t)$$

... reduce to paths for Schur polynomials.



$$P_\varphi(0,0)P_\varphi(0,0)$$



$2\omega_1 + 2\omega_2$
$3\omega_1$
$3\omega_1$
φ
0