A LITTLEWOOD-RICHARDSON RULE FOR MACDONALD POLYNOMIALS UW-MADISON COMBINATORICS SEMINAR SEPTEMBER 28 2009

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Contents

1. What are Macdonald polynomials?

They are orthogonal multivariate Laurent polynomials associated to root systems with 2 parameters q, t.

Fix

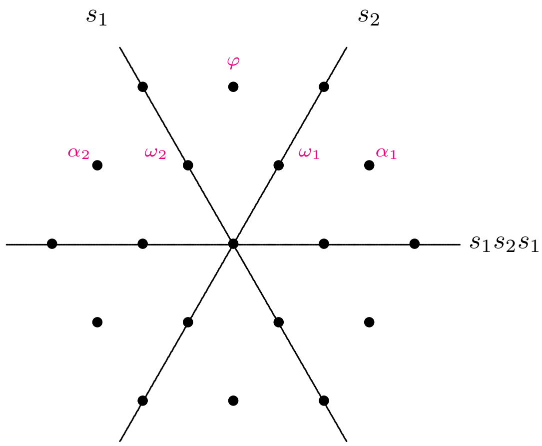
R	a root system
W_0	Weyl group of R
$L, L^\vee$	weight, and coweight lattice of R

If $L = \sum_{i=1}^n \mathbb{Z}\omega_i$, then for $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n \in L$, write

$$X^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

Example of the A_2 root system: Let $V^* = \text{span}\{e_1, e_2, e_3\}/\text{span}\{e_1+e_2+e_3\}$ be a two-dimensional vector space over $\mathbb{R}$. Let $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \varphi = e_1 - e_3$.

$$\begin{aligned} R &= \{\pm\alpha_1, \pm\alpha_2, \pm\varphi\} \\ W_0 &= \mathfrak{S}_3 = \langle s_1, s_2 \mid s_1^2 = 1 = s^2, s_1s_2s_1 = s_2s_1s_2\rangle \\ L \cong L^\vee &= \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \end{aligned}$$



There are two kinds. For $\lambda \in L$:

$$\text{Nonsymmetric: } E_\lambda(X;q,t)$$
$$\text{Symmetric: } P_\lambda(X;q,t)$$

Let $K = \mathbb{Q}(q, t^{1/2})$. Then

$\{E_\lambda : \lambda \in L\}$ is a basis for $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$
$\{P_\lambda : \lambda \in L_+\}$ is a basis for $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_0}$ (ie. symmetric polynomials)

$\{E_\lambda \mid \lambda \in L\}$ is a family of polynomials such that

- $(E_\lambda, E_\mu) = 0$ if $\mu \neq \lambda$,
- $E_\lambda = X^\lambda + \sum_{\mu<\lambda} a_\mu(q,t) X^\mu$ for $\mu < \lambda$ in the Bruhat order.

Where, if we write a generic polynomial as $g = \sum_\lambda g_\lambda(q,t)X^\lambda$, and set $g^* = \sum_\lambda g_\lambda(q^{-1}, t^{-1})X^{-\lambda}$, then

$$(f,g) = [X^0] f g^* \Delta,$$

and (for simplicity we assume $t = q^k$, $k \in \mathbb{Z}_{>0}$; in general, Δ is an infinite sum),

$$\Delta = \prod_{\alpha \in R_+} (1-X^\alpha)(1-qX^\alpha)\cdots(1-q^{k-1}X^\alpha) \prod_{\alpha \in R_-} (1-qX^\alpha)(1-q^2X^\alpha)\cdots(1-q^kX^\alpha).$$

$\{P_\lambda \mid \lambda \in L_+\}$ is a family of polynomials such that

- $(P_\lambda, P_\mu)' = 0$ if $\mu \neq \lambda$,
- $P_\lambda = M_\lambda + \sum_{\mu<\lambda} b_\mu(q,t) M_\mu$ for $\mu < \lambda$ in the Bruhat order, $M_\lambda = \sum_{\nu \in W_0\lambda} X^\nu$ is the orbit sum.

Where, if we set $\bar{g} = \sum_\lambda g_\lambda(q,t) X^{-\lambda}$, then

$$(f,g)' = \frac{1}{|W_0|}[X^0] f \bar{g} \nabla,$$

and (again assume $t = q^k$, $k \in \mathbb{Z}_{>0}$),

$$\nabla = \prod_{\alpha \in R} (1-X^\alpha)(1-qX^\alpha)\cdots(1-q^{k-1}X^\alpha).$$

The symmetric Macdonald polynomials specialize to many known families of symmetric polynomials, including spherical functions on compact Lie groups and p-adic Lie groups, and Weyl characters of simple Lie algebras. (Schur polynomials, for example, at $q = t$ Type A).

Computation of these polynomials using these definitions is hard!

2. Double affine Hecke algebras and intertwiners

Cherednik created the double affine Hecke algebra $\tilde{\mathcal{H}}$ and used it to prove some hard conjectures made by Macdonald, including the evaluation conjecture, duality conjecture, and constant term conjecture (special case of norm conjecture), where in the case $t = q^k$, is the statement that

$$[X^0]\nabla = \prod_{\alpha\in R_+}\prod_{i=1}^{k-1}\frac{1-q^{\langle k\rho,\alpha^\vee\rangle+i}}{1-q^{\langle k\rho,\alpha^\vee\rangle-i}}.$$

Intertwining operators of the DAHA are "creation operators" for E_λ. $\tilde{H}$ is a quotient of the double affine braid group $\tilde{\mathcal{B}}$.

Definitions.

(think S_{n+1}) finite Weyl group W_0 is generated by $s_1,\ldots,s_n$ satisfying braid relations and $s_i^2 = 1$,

affine Weyl group $W = W_0 \ltimes L^\vee$ acts as group of reflections and translations on $L^\vee$,

braid group B_0 is generated by $T_1,\ldots,T_n$ satisfying braid relations,

a caveat: if we replace L by the root lattice Q, some things are easier to describe,
ie $W = W_0 \ltimes Q^\vee$ is generated by $s_0,\ldots,s_n$ satisfying braid relations and $s_i^2 = 1$,
where $s_0 = t(\varphi^\vee)s_\varphi$.

affine braid group B is generated by $B_0, Y^{L^\vee}$ where $Y^{L^\vee}$ is an abelian group that is an analogue of $L^\vee$,
and we have the element $T_0 = Y^{\varphi^\vee}T_{s_\varphi}^{-1}$

double affine braid group $\tilde{B}$ is generated by $X^L, B_0, Y^{L^\vee}, q$
where X^L is an abelian group, q is a central element, and we have relations

$T_i^{-1}Y^{\lambda^\vee} = Y^{\lambda^\vee}T_i^{-1}$ if $\langle\lambda^\vee,\alpha_i\rangle = 0$ for $1\le i\le n$,

$T_i^{-1}Y^{\lambda^\vee}T_i^{-1} = Y^{s_i\lambda^\vee}$ if $\langle\lambda^\vee,\alpha_i\rangle = 1$ for $1\le i\le n$,

$T_iX^\mu = X^\mu T_i$ if $\langle\alpha_i^\vee,\mu\rangle = 0$ for $0\le i\le n$,

$T_iX^\mu T_i = X^{s_i\mu}$ if $\langle\alpha_i^\vee,\mu\rangle = 1$ for $0\le i\le n$.

The double affine Hecke algebra $\tilde{H}$ is the quotient of $K\tilde{\mathcal{B}}$ by the ideal generated by

$$T_i^2 = (t^{1/2}-t^{-1/2})T_i + 1,\quad 0\le i\le n.$$

A basis for $\tilde{H}$ is

$$\{q^kX^\mu T_wY^{\lambda^\vee} \mid k\in\mathbb{Z}, \mu\in L, w\in W_0, \lambda^\vee\in L^\vee\}.$$

The polynomial representation of $\tilde{H}$ is $K[X]\mathbf{1}$ where

$$X^\mu\mathbf{1} = X^\mu\mathbf{1},\quad T_i\mathbf{1} = t^{1/2}\mathbf{1},\quad Y^{\lambda^\vee}\mathbf{1} = t^{\mathrm{ht}(\lambda^\vee)}\mathbf{1}.$$

The height is defined by

$$\mathrm{ht}(\lambda^\vee) = \left\langle \lambda^\vee, \tfrac{1}{2} \sum_{\alpha\in R_+} \alpha \right\rangle.$$

For example, $\mathrm{ht}(\alpha_1^\vee) = 1 = \mathrm{ht}(\alpha_2^\vee)$, $\mathrm{ht}(\varphi^\vee) = 2$.

The intertwiners are

$$\tau_i^\vee = T_i + \tfrac{t^{-1/2}-t^{1/2}}{1-Y^{-\alpha_i^\vee}}, \qquad \text{for } 1 \le i \le n,$$
$$\tau_0^\vee = (X^\varphi T_{s_\varphi})^{-1} + \tfrac{t^{-1/2}-t^{1/2}}{1-qY^{\varphi^\vee}}, \quad \text{(in Type A)}.$$

Let $\lambda \in L$. Note that $W/W_0 \cong L$. Let

$$m_\lambda \text{ be the shortest element in the coset } t(\lambda)W_0.$$

Cherednik showed that if $m_\lambda = s_{i_1}\cdots s_{i_r} \in W$ is a reduced word, then

$$E_\lambda \mathbf{1} = \tau_{i_1}^\vee \cdots \tau_{i_r}^\vee \mathbf{1},$$
$$P_\lambda \mathbf{1} = \mathbf{1}_0 \tau_{i_1}^\vee \cdots \tau_{i_r}^\vee \mathbf{1},$$

where $\mathbf{1}_0 = \sum_{w\in W_0} t^{-\ell(w_0 w)/2} T_w$.

For example, choose $\lambda = -\alpha_2$. Then $m_\lambda = s_2 s_1 s_0^\vee$, so

$$E_{-\alpha_2}\mathbf{1} = \tau_2^\vee \tau_1^\vee \tau_0^\vee \mathbf{1}.$$

The problem in computing this is we don't know how to commute X and Y easily.

3. Alcove walks

Alcove walks make a good combinatorial model for the expansion of products of intertwiners.

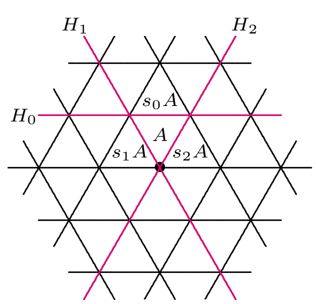
For the A_2 system, the affine coroots are

$$S^\vee = \{\alpha^\vee + r \mid \alpha^\vee \in R^\vee, r \in \mathbb{Z}\}.$$

Think of them as polynomial functions of degree ≤ 1 on $L_\mathbb{R} = \mathbb{R}\omega_1 + \mathbb{R}\omega_2$. Each affine coroot $a^\vee \in S$ defines a hyperplane in $L_\mathbb{R}$

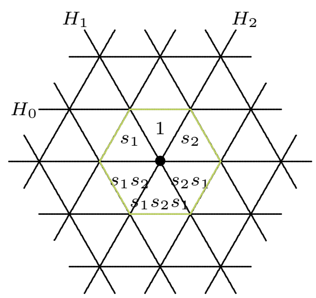
$$H_{a^\vee} = {a^\vee}^{-1}(0).$$

These cut up the vector space $L_\mathbb{R}$ into regions called alcoves.



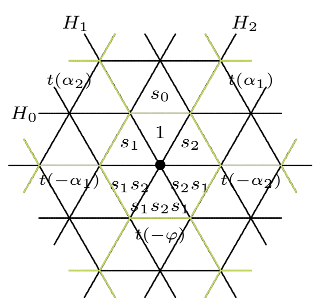
The alcoves are in bijection with elements of $W^\vee = Q \rtimes W_0$:

$$\begin{aligned} Q \rtimes W_0 = W^\vee &\longleftrightarrow \text{alcoves} \\ W_0 &\longleftrightarrow \text{alcoves in hexagon centered at } 0 \end{aligned}$$



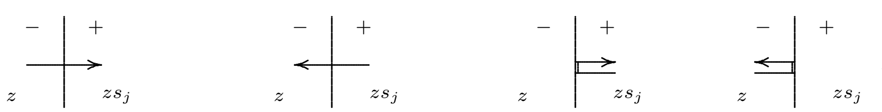
$S_3 \longleftrightarrow$ alcoves in 0-hexagon

$$\begin{aligned} Q \rtimes W_0 = W^\vee &\longleftrightarrow \text{alcoves} \\ W_0 &\longleftrightarrow \text{alcoves in hexagon centered at } 0 \\ t(\mu)W_0 &\longleftrightarrow \text{alcoves in hexagon centered at } \mu \end{aligned}$$



$t(\mu)W_0 \longleftrightarrow$ shifted hexagons

Let $w = s_{i_1} \cdots s_{i_r} \in W^\vee$. An alcove walk of type $i_1, \ldots, i_r$ is a sequence of r steps where the jth step is one of



positive j-crossing, negative j-crossing, positive j-fold, negative j-fold

Theorem 3.1. *(Ram, Y)*

$$E_\lambda = \sum_{p \in B(m_\mu)} t^{\frac{1}{2}\ell(d(p))} f_p X^{\mathtt{wt}(p)}$$

is a sum of walks of type m_μ starting in the fundamental alcove, where the folding coefficient is

$$f_p = \prod_{kth\ step\ a\ fold} \frac{t^{-1/2} - t^{1/2}}{1 - q^{\mathrm{sh}(b_k)} t^{\mathrm{ht}(b_k)}} \prod_{kth\ step\ a\ \text{-ve}\ fold} q^{\mathrm{sh}(b_k)} t^{\mathrm{ht}(b_k)},$$

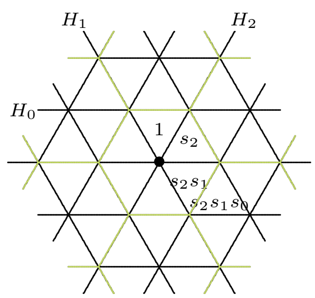
b_k is the affine coroot that defines the hyperplane crossed by the kth last step of the inverse walk m_μ^{-1}, and $\ell(d(p))$ is the distance between the endpoint of the path, and the top alcove in the W_0-coset it ends in.

Similarly,

$$P_\lambda = \sum_{p \in B(m_\mu)} t^{-\frac{1}{2}\ell(\iota(p) w_0)} t^{\frac{1}{2}\ell(d(p))} f_p X^{\mathtt{wt}(p)}$$

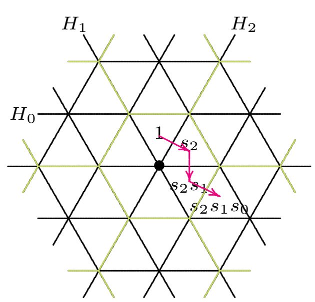
is a sum over paths starting in any alcove in the hexagon centered at zero.

For example, $m_{-\alpha_2} = s_2 s_1 s_0$. Since each step can be a fold or a crossing, there are $2^3 = 8$ paths of this type.



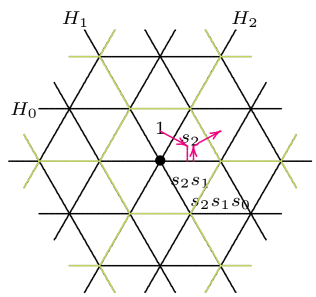
So the path formula for $E_{-\alpha_2}$ has eight terms. The one with no folds corresponds to

$$t^{1/2} X^{-\alpha_2}$$

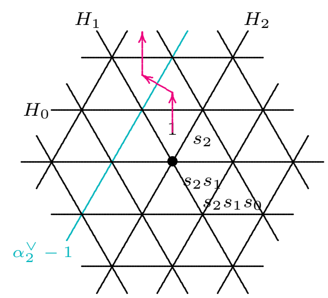


the one whose 2nd step is folded, the other two are straight corresponds to

$$t \frac{t^{-1/2} - t^{1/2}}{1 - qt} X^{\alpha_1}.$$



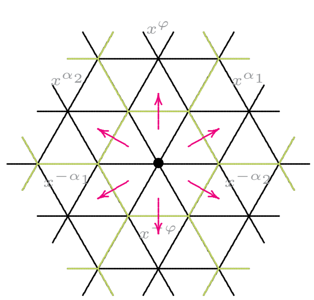
How do we find the folding coefficient in this case? The inverse walk is $s_0^\vee s_1 s_2$, and the second last hyperplane crossed by the inverse walk is defined by the coroot $\alpha_2^\vee - 1$. This has shift 1, and height 1. The fold in p is positive. The final position is two alcoves away from the top alcove in the $t(\alpha_1)$ coset.



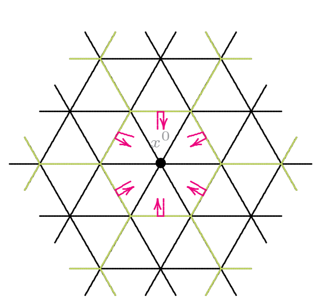
The proof makes use of Cherednik's intertwiner formula, and is by induction on the length of $m_\mu \in W^\vee$. Roughly, in $\tau_i^\vee$, the $T_i^\vee$ term corresponds to a step, and the $\frac{t^{-1/2}-t^{-1/2}}{1-Y^{-\alpha_i^\vee}}$ term corresponds to a fold. While X and Y commutation relations are difficult, $\tau_i^\vee$ and Y commutation relations are easy, and walks are an effective way to capture this.

Example

$$P_\varphi(q,t) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + (2+t+q+2qt)\frac{1-t}{1-qt^2}x^0$$



M_φ



$(2+t+q+2qt)\frac{1-t}{1-qt^2}$

4. A Littlewood-Richardson rule

Theorem 4.1.

$$E_\mu P_\lambda = \sum_{p \in \Gamma(m_\mu^{-1})} b_p f_p f_p n_p E_{\mathtt{wt}(p)},$$
$$P_\mu P_\lambda = \sum_{p \in \Gamma(m_\mu^{-1})} b_p e_p f_p f_p n_p P_{\mathtt{wt}(p)},$$

are sums over paths of type m_μ^{-1} starting in the alcoves of the hexagon centered at $-w_0\lambda$, and is contained in the dominant chamber, and whose folds are bi-coloured.

$$b_p = \prod_{a^\vee \in m_\lambda^{-1}\mathcal{L}(\mathrm{i}(p))} \frac{t^{1/2} - t^{-1/2} q^{\mathrm{sh}(a^\vee)} t^{\mathrm{ht}(a^\vee)}}{1 - q^{\mathrm{sh}(a^\vee)} t^{\mathrm{ht}(a^\vee)}},$$

$$f_p = \prod_{k \in \phi(p)} \frac{t^{1/2} - t^{-1/2}}{1 - q^{\mathrm{sh}(b_k^\vee)} t^{\mathrm{ht}(b_k^\vee)}} \prod_{k \in \phi^-(p)} q^{\mathrm{sh}(b_k^\vee)} t^{\mathrm{ht}(b_k^\vee)},$$

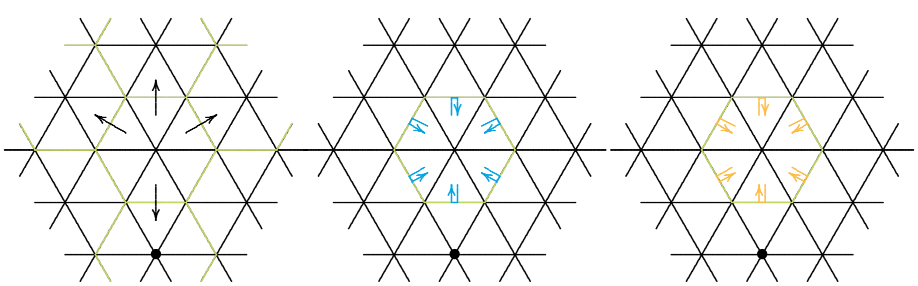
$$f_p = \prod_{k \in \phi(p)'} \frac{t^{1/2} - t^{-1/2}}{1 - q^{\mathrm{sh}(c_k^\vee)} t^{\mathrm{ht}(c_k^\vee)}} \prod_{k \in \phi(p)', r-k+1 \in \xi^+(p)} q^{\mathrm{sh}(c_k^\vee)} t^{\mathrm{ht}(c_k^\vee)},$$

$$n_p = \prod_{j \in \xi^-(p)} \frac{1 - q^{\mathrm{sh}(h_j^\vee)} t^{\mathrm{ht}(h_j^\vee)-1}}{1 - q^{\mathrm{sh}(h_j^\vee)} t^{\mathrm{ht}(h_j^\vee)}} \frac{1 - q^{\mathrm{sh}(h_j^\vee)} t^{\mathrm{ht}(h_j^\vee)+1}}{1 - q^{\mathrm{sh}(h_j^\vee)} t^{\mathrm{ht}(h_j^\vee)}},$$

$$e_p = \prod_{a^\vee \in m^{-1}_{-w_0 \mathtt{wt}(p)} \mathcal{L}(\mathtt{c}(p)^{-1})} \frac{t^{-1/2} - t^{1/2} q^{\mathrm{sh}(a^\vee)} t^{\mathrm{ht}(a^\vee)}}{1 - q^{\mathrm{sh}(a^\vee)} t^{\mathrm{ht}(a^\vee)}}.$$

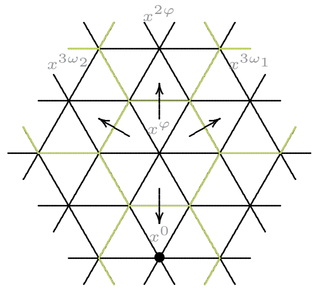
Example

$$P_\varphi = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + (2 + t + q + 2qt)\frac{1-t}{1-qt^2}.$$

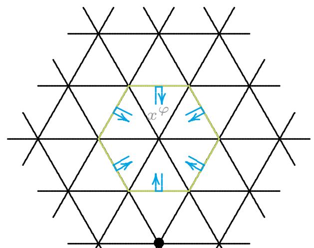


$$P_\varphi(q,t) P_\varphi(q,t)$$

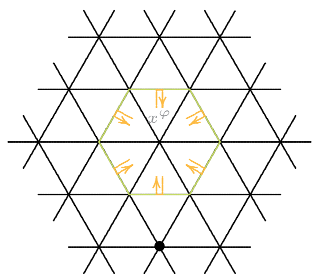
$P_\varphi(q,t) P_\varphi(q,t) =$



$$P_{2\varphi} + \frac{1-q}{1-qt} t^{1/2} P_{3\omega_1} + \frac{1-q}{1-qt} t^{1/2} P_{3\omega_2} + \frac{1-q}{1-qt} \frac{1-q^2t}{1-q^2t^2} \frac{1-q}{1-qt} \frac{1-qt}{1-qt^2} \frac{1-qt^3}{1-qt^2} t^{3/2} P_0 + \cdots$$



$$\frac{(t^{1/2}-t^{-1/2})q^3t^2}{1-q^3t^2} \frac{t^{-1/2}-t^{1/2}qt}{1-qt} \frac{t^{-1/2}-t^{1/2}q^2t^2}{1-q^2t^2} \frac{t^{-1/2}-t^{1/2}qt}{1-qt} P_\varphi + 2\frac{t^{1/2}-t^{-1/2}qt}{1-qt} \frac{(t^{1/2}-t^{-1/2})q^2t}{1-q^2t} \frac{t^{-1/2}-t^{1/2}qt}{1-qt} \frac{t^{-1/2}-t^{1/2}q^2t^2}{1-q^2t^2} P_\varphi$$
$$2\frac{t^{1/2}-t^{-1/2}qt}{1-qt} \frac{t^{1/2}-t^{-1/2}q^2t^2}{1-q^2t^2} \frac{t^{-1/2}-t^{1/2}}{1-t} \frac{t^{-1/2}-t^{1/2}qt}{1-qt} P_\varphi + \frac{t^{1/2}-t^{-1/2}qt}{1-qt} \frac{t^{1/2}-t^{-1/2}qt}{1-qt} \frac{t^{1/2}-t^{-1/2}q^2t^2}{1-q^2t^2} \frac{t^{-1/2}-t^{1/2}}{1-qt^2} P_\varphi + \cdots$$



$$q\frac{1-t}{1-qt^2}(1+2t+2t^2+t^3)P_\varphi.$$

Recall that $\{P_\lambda \mid \lambda \in L_+\}$ is a basis for symmetric functions, and at $q = t$, the type A ones reduce to Schur functions, which are characters of irreducible representations of GL_n. So this is a generalization of the classical Littlewood-Richardson rule for the direct sum decomposition of the tensor product of two irreducible representations of GL_n.

$$V(\mu) \otimes V(\lambda) = \bigoplus_\nu c^\nu_{\mu\lambda} V(\nu).$$

The original form of the Littlewood-Richardson rule was a combinatorial formula given in terms of fillings of tableaux.

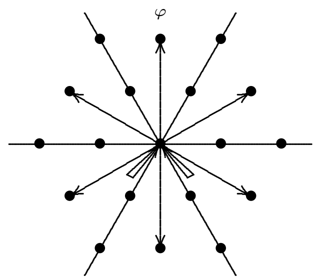
There is another combinatorial formula for the coefficients $c^\nu_{\mu\lambda}$ given in terms of Littelmann paths.

Example

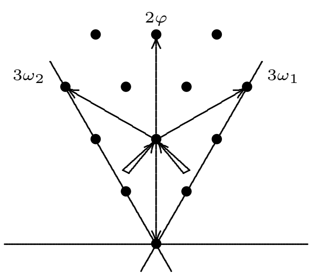
The Littelmann path formula for the Schur function s_φ is

$$s_\varphi = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\varphi} + x^{-\alpha_1} + x^{-\alpha_2} + 2x^0,$$

with terms corresponding to the paths



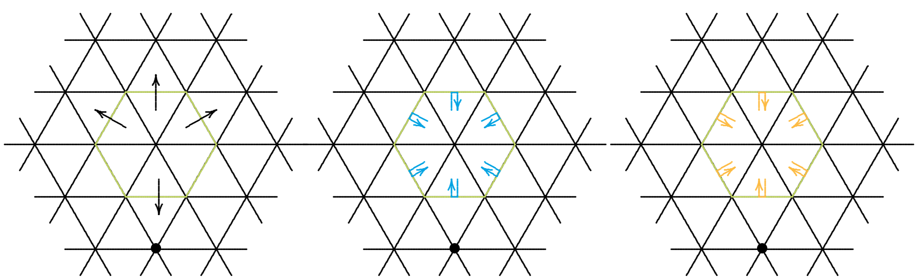
And the Littelmann path formula for the product $s_\varphi s_\varphi$ involves taking the paths for s_φ, and shifting them to start at φ, keeping only the paths that are contained in the dominant chamber



so

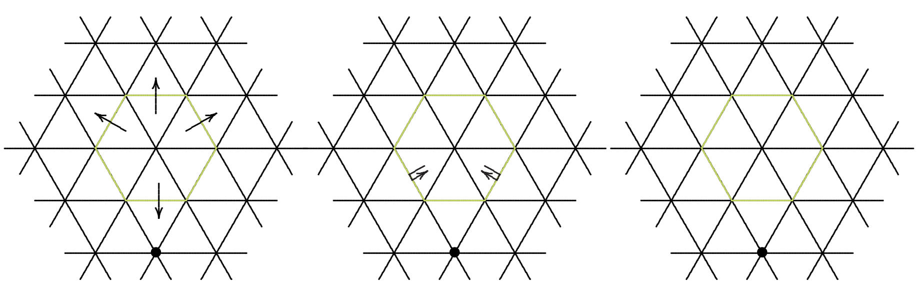
$$s_\varphi s_\varphi = s_{2\varphi} + s_{3\omega_1} + s_{3\omega_2} + 2s_{\omega_1+\omega_2} + s_0.$$

Thus bi-coloured alcove walks for Macdonald polynomials ...



$P_\varphi(q,t)P_\varphi(q,t)$

... reduce to paths for Schur polynomials.



$P_\varphi(0,0)P_\varphi(0,0)$

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