A Littlewood-Richardson rule for Macdonald polynomials

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$$s_{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2 x_1 x_2 x_3$$

$$s_{\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline\end{array}} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$$

$$+ \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline\end{array}$$

$$= x_1^3 + x_2^3 + x_3^3 + x_1 x_2 x_3$$

$$+ x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

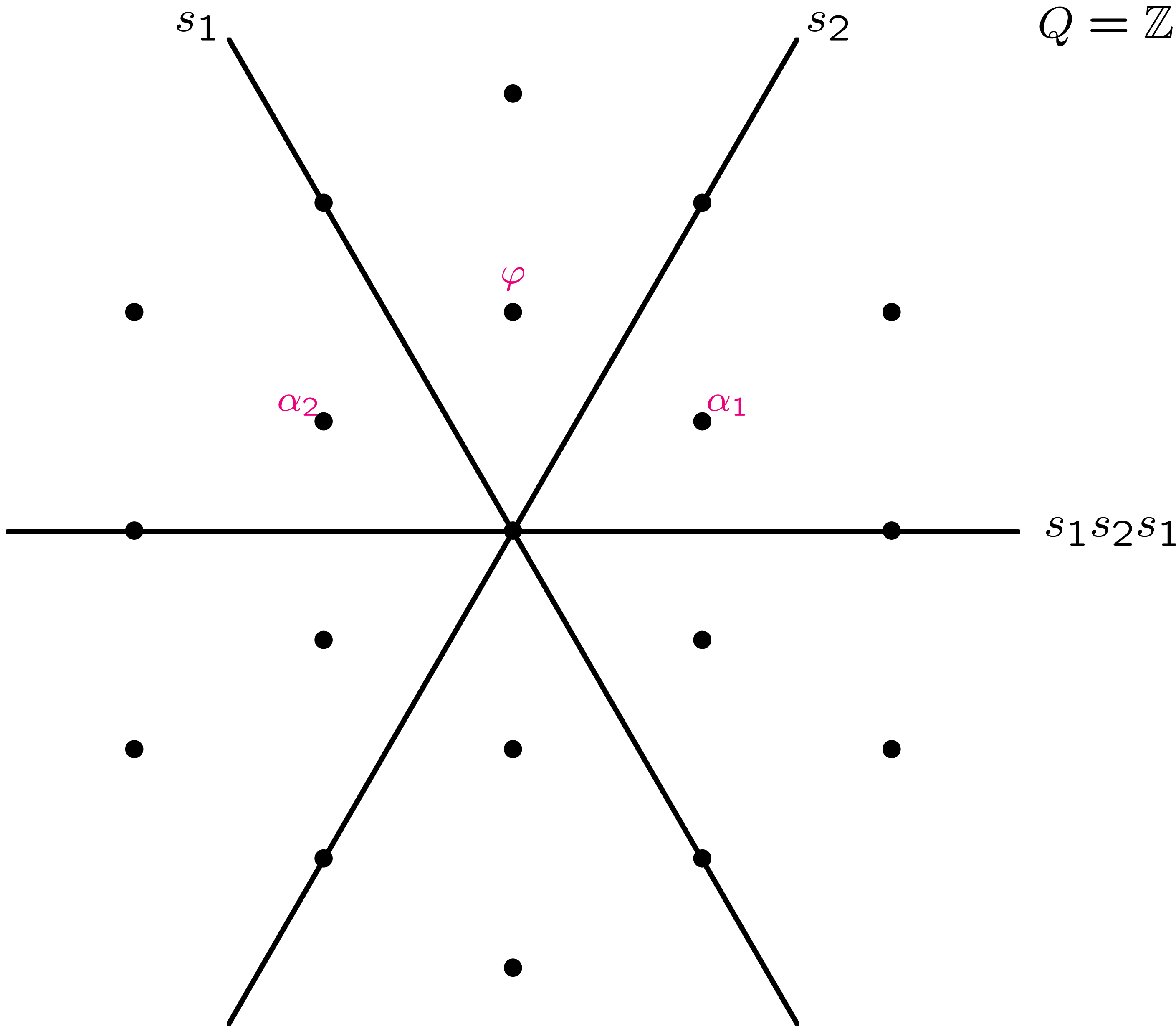
$$s_{\yng(2,1)}\, s_{\yng(2,1)} = s_{\yng(4,2)} + s_{\yng(4,1,1)} + s_{\yng(3,3)} + 2s_{\yng(3,2,1)} + s_{\yng(2,2,2)}$$

$$\young(\ \ 11,\ 2) \qquad \young(\ \ 11,\ ,2) \qquad \young(\ \ 1,\ 12) \qquad \young(\ \ 1,\ 1,2) \qquad \young(\ \ 1,\ 2,1) \qquad \young(\ \ ,\ 1,12)$$

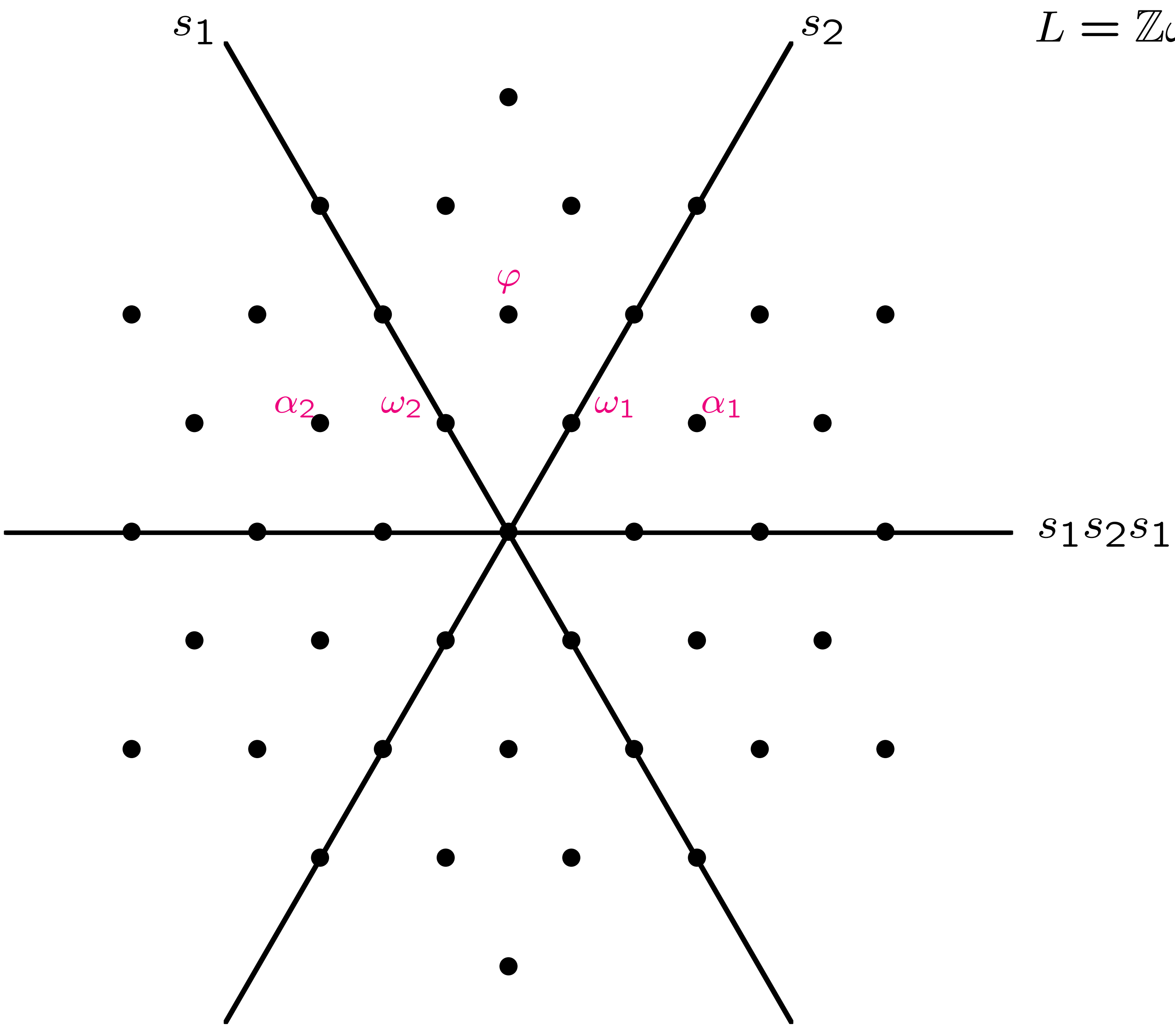
$$s_{\yng(3)}\, s_{\yng(2,1)} = s_{\yng(5,1)} + s_{\yng(4,2)} + s_{\yng(4,1,1)} + s_{\yng(3,2,1)}$$

$$\young(\ \ 111,\ ) \qquad \young(\ \ 11,\ 1) \qquad \young(\ \ 11,\ ,1) \qquad \young(\ \ 1,\ 1,1)$$

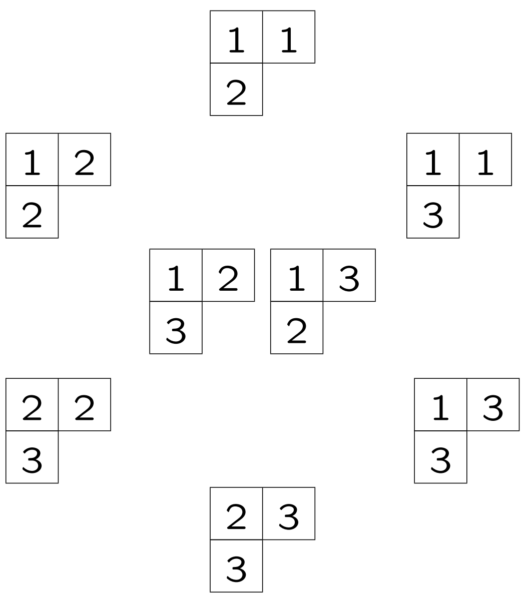
$Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$



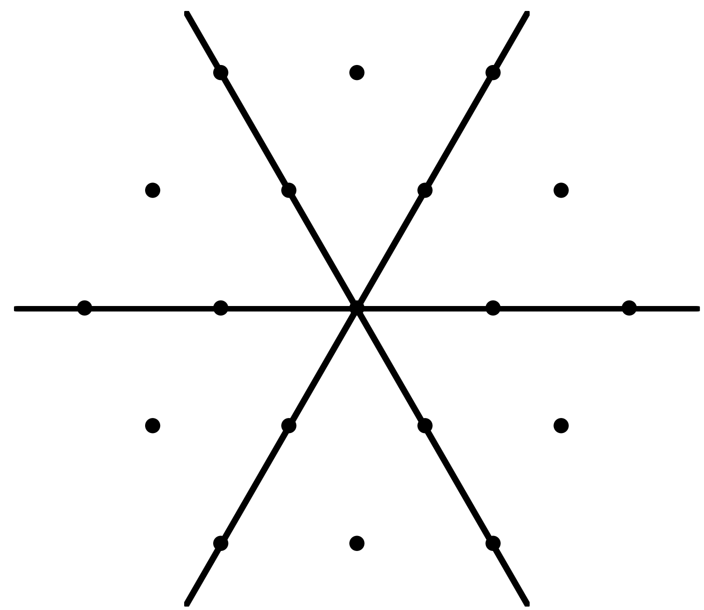
$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

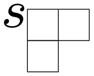


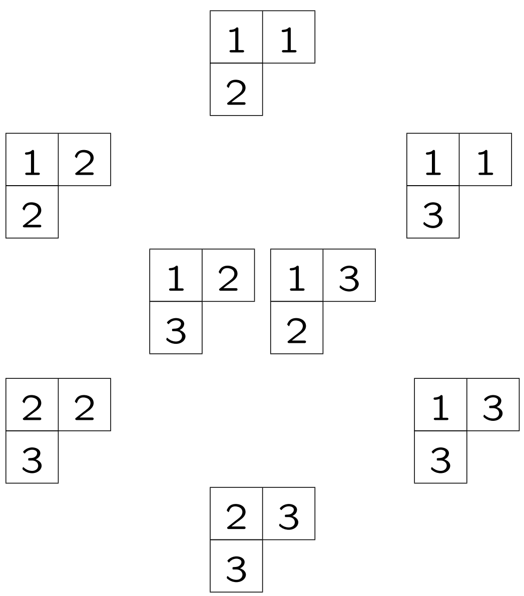
$$s_{\yng(2,1)} \quad \longleftrightarrow \quad s_{\omega_1+\omega_2}$$



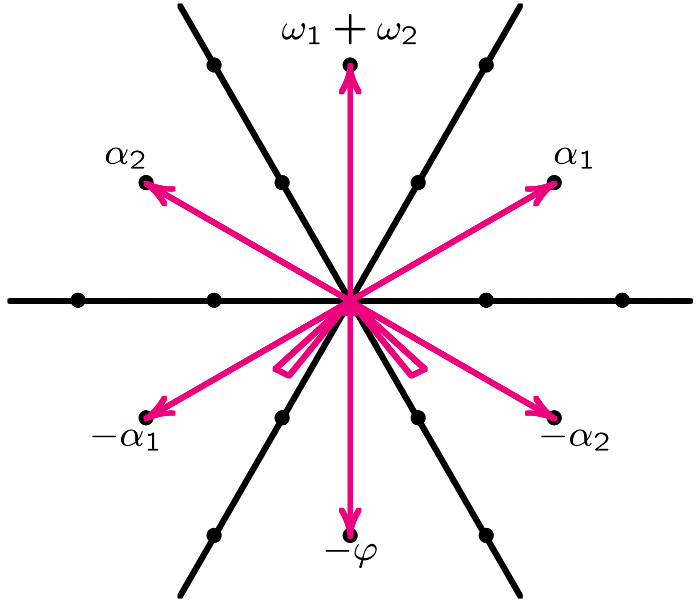
$\longleftrightarrow$



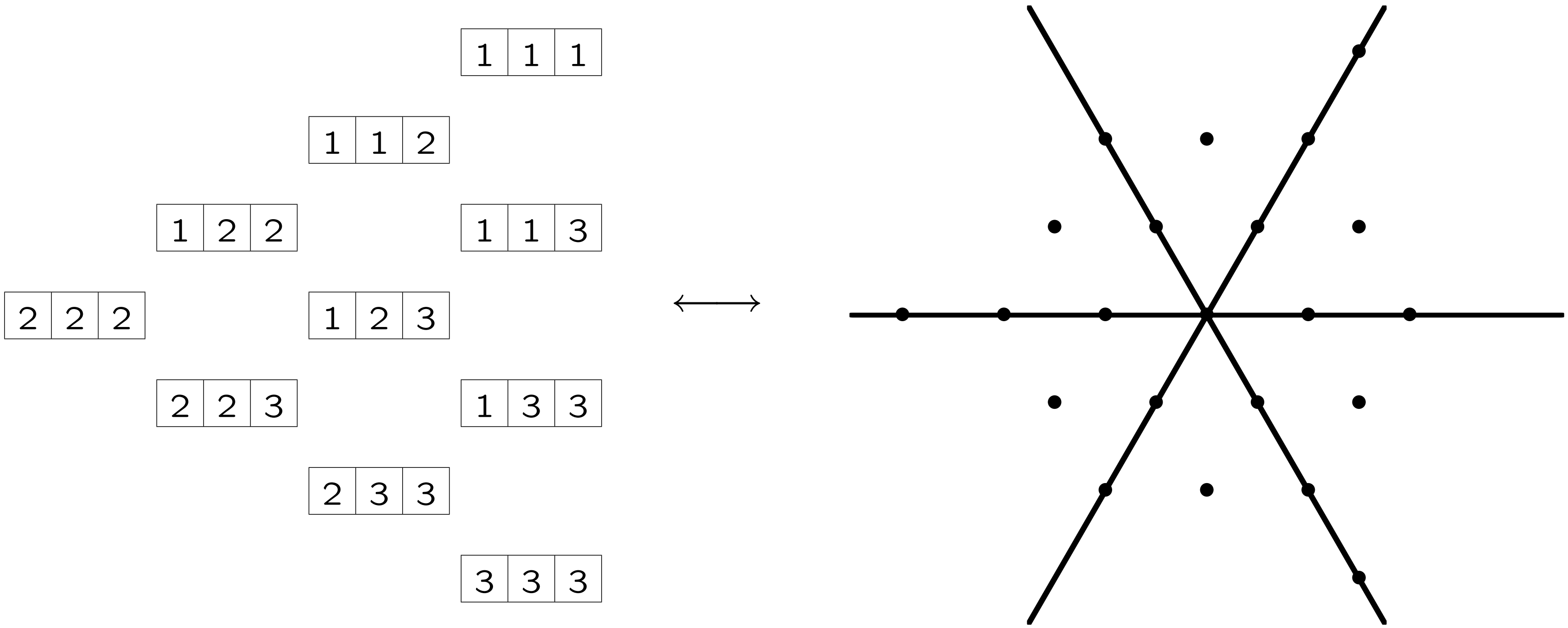
s  $\longleftrightarrow \quad s_{\omega_1+\omega_2} = s_{\varphi}$



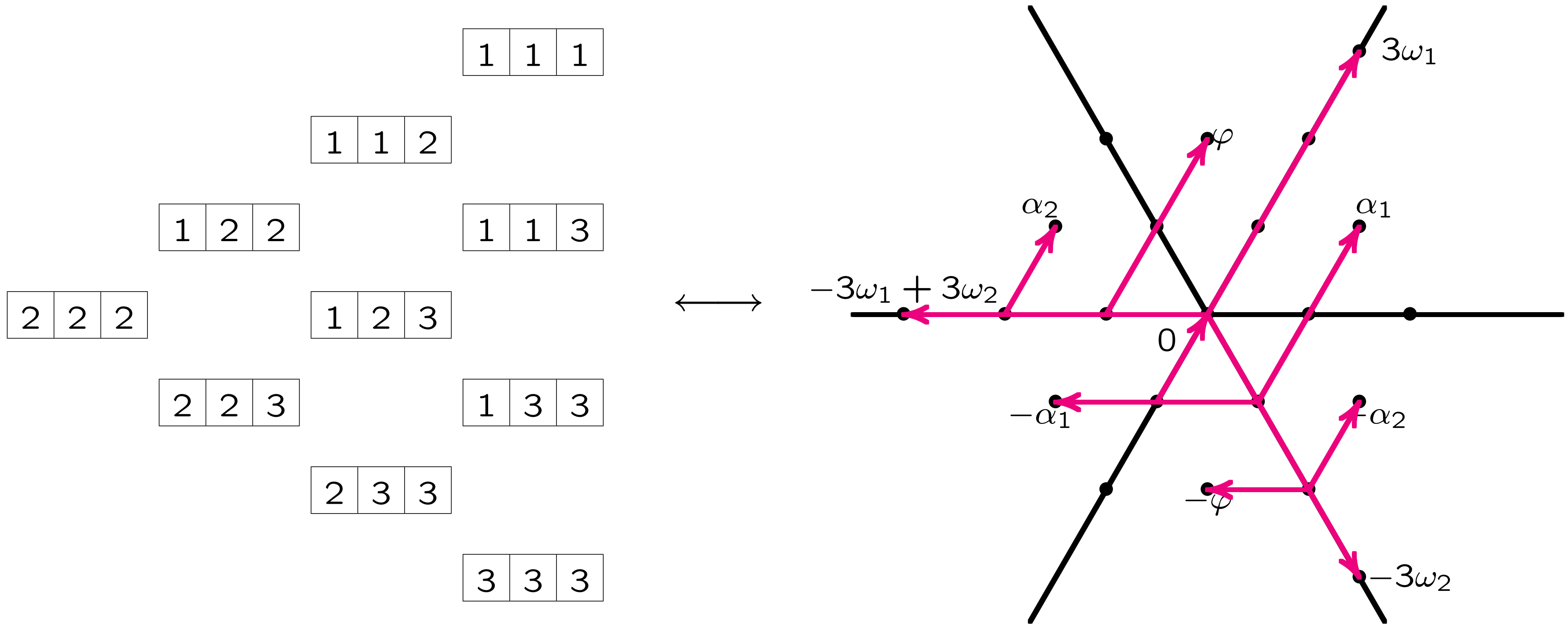
$\longleftrightarrow$



$$s_{\square\square\square} = s_{3\omega_1}$$



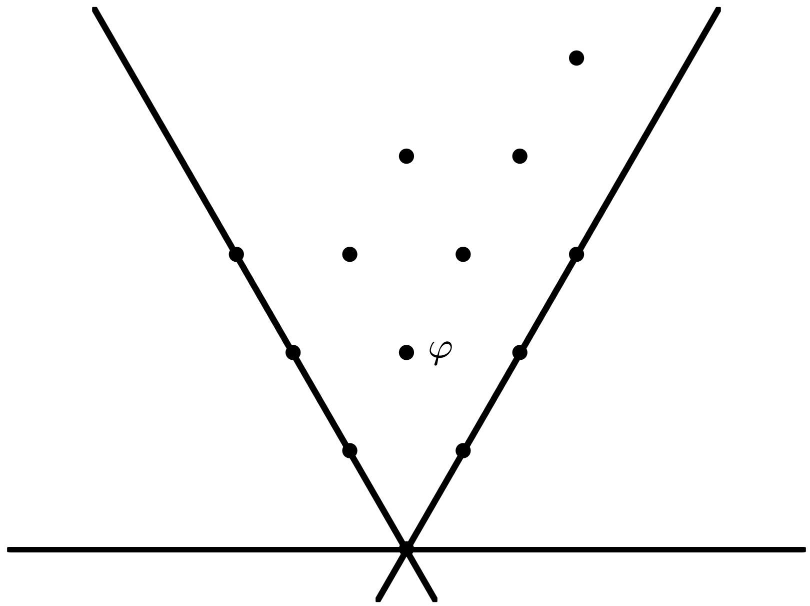
$$s_{\Box\Box\Box} = s_{3\omega_1}$$



5-a

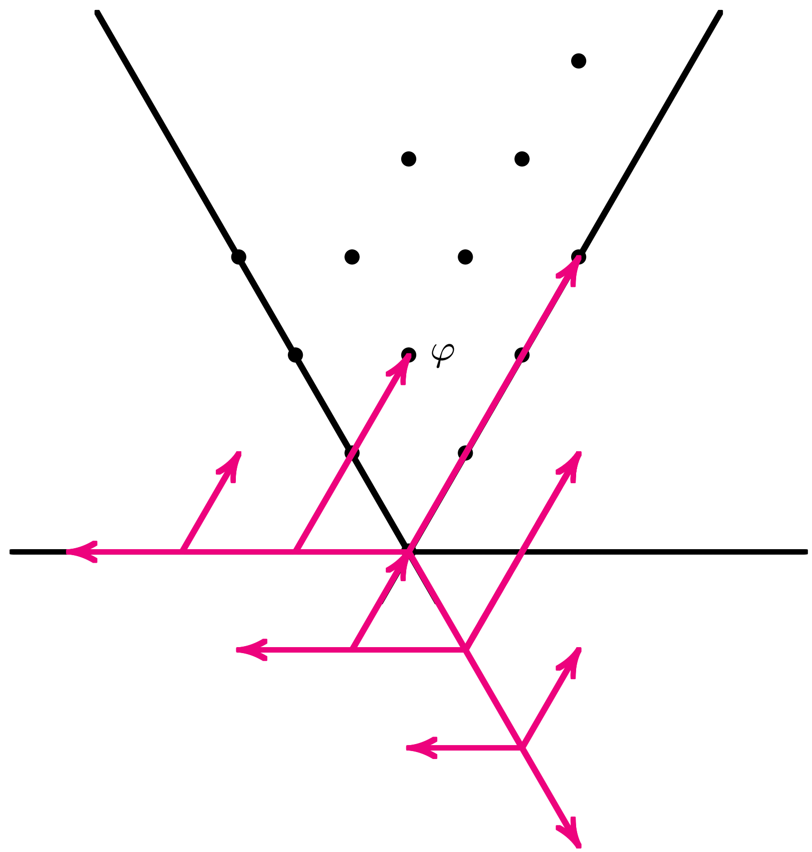
$$s_{3\omega_1} s_{\varphi} \quad = \quad s_{4\omega_1+\omega_2} + s_{2\varphi} + s_{3\omega_1} + s_{\varphi}$$

$$s_{(3)} s_{(2,1)} \quad = \quad s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + s_{(3,2,1)}$$



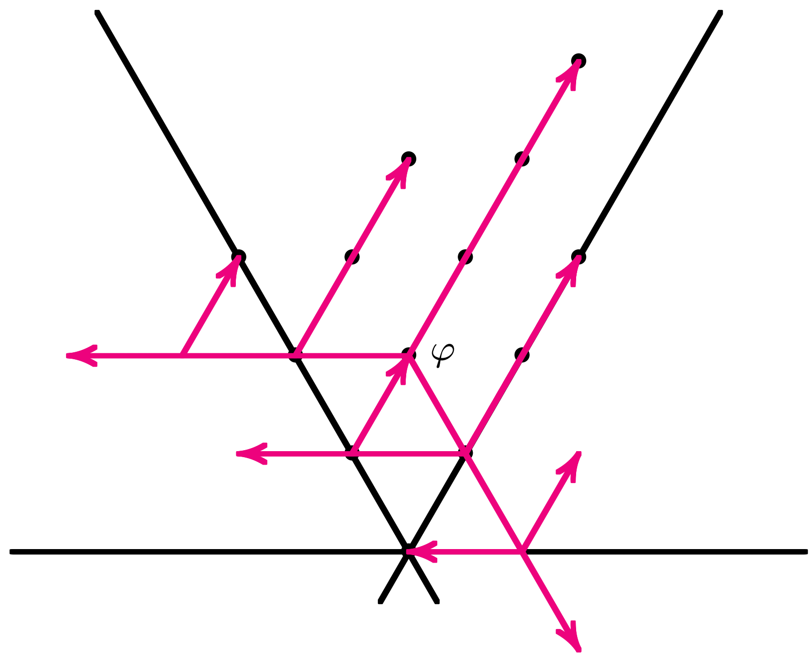
$$s_{3\omega_1} s_{\varphi} = s_{4\omega_1+\omega_2} + s_{2\varphi} + s_{3\omega_1} + s_{\varphi}$$

$$s_{(3)} s_{(2,1)} = s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + s_{(3,2,1)}$$



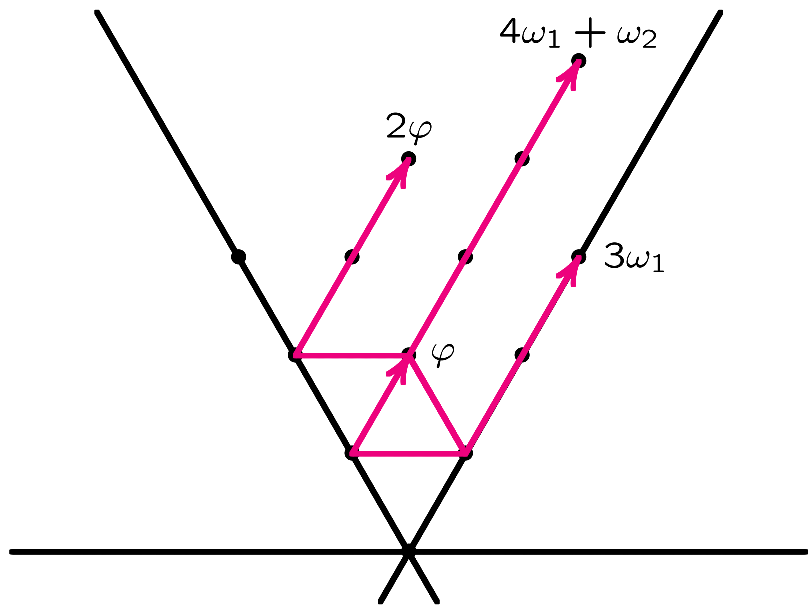
$$s_{3\omega_1} s_{\varphi} = s_{4\omega_1+\omega_2} + s_{2\varphi} + s_{3\omega_1} + s_{\varphi}$$

$$s_{(3)} s_{(2,1)} = s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + s_{(3,2,1)}$$

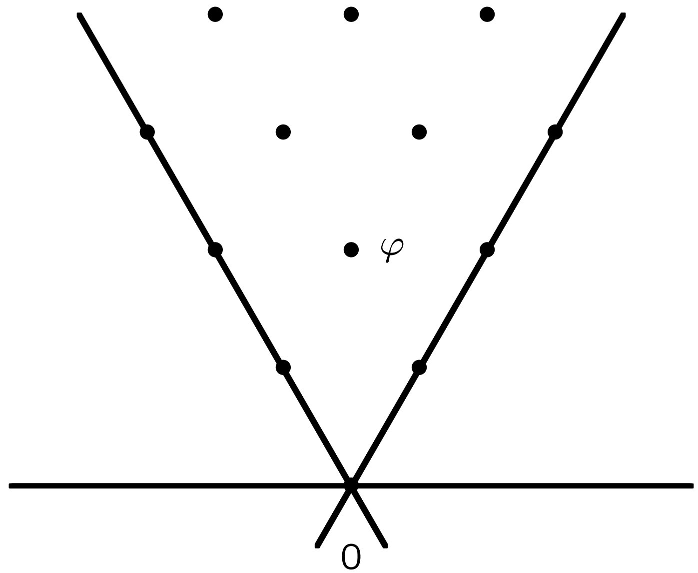


$$s_{3\omega_1} s_{\varphi} = s_{4\omega_1+\omega_2} + s_{2\varphi} + s_{3\omega_1} + s_{\varphi}$$

$$s_{(3)} s_{(2,1)} = s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + s_{(3,2,1)}$$

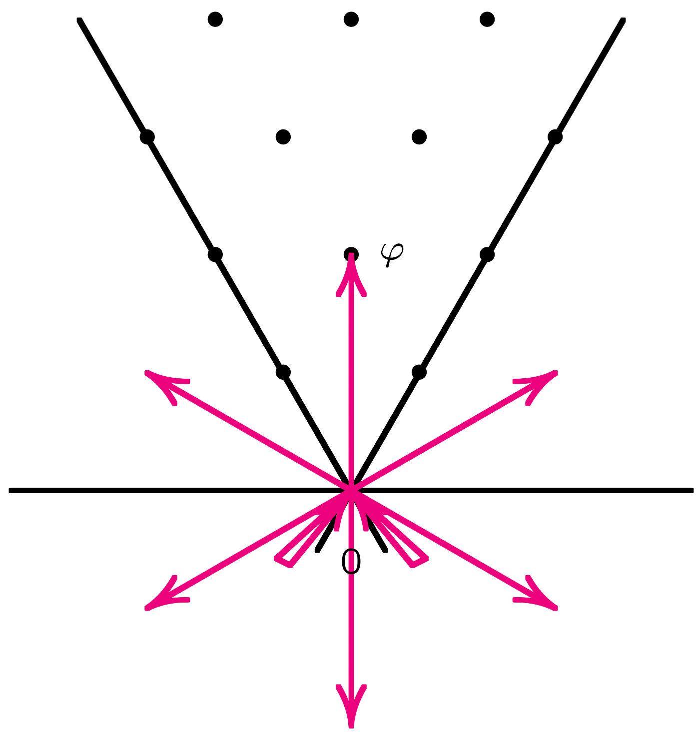


$$
\begin{aligned}
s_\varphi s_\varphi &= s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_\varphi + s_0 \\
s_{(2,1)} s_{(2,1)} &= s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(2,2,2)}
\end{aligned}
$$



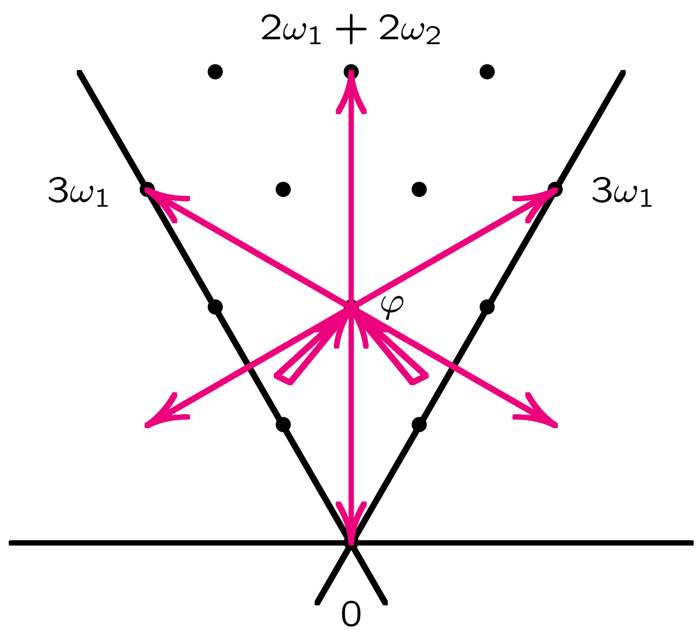
$$s_\varphi s_\varphi = s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_\varphi + s_0$$

$$s_{\ydiagram{2,1}} s_{\ydiagram{2,1}} = s_{\ydiagram{4,2}} + s_{\ydiagram{4,1,1}} + s_{\ydiagram{3,3}} + 2s_{\ydiagram{3,2,1}} + s_{\ydiagram{2,2,2}}$$



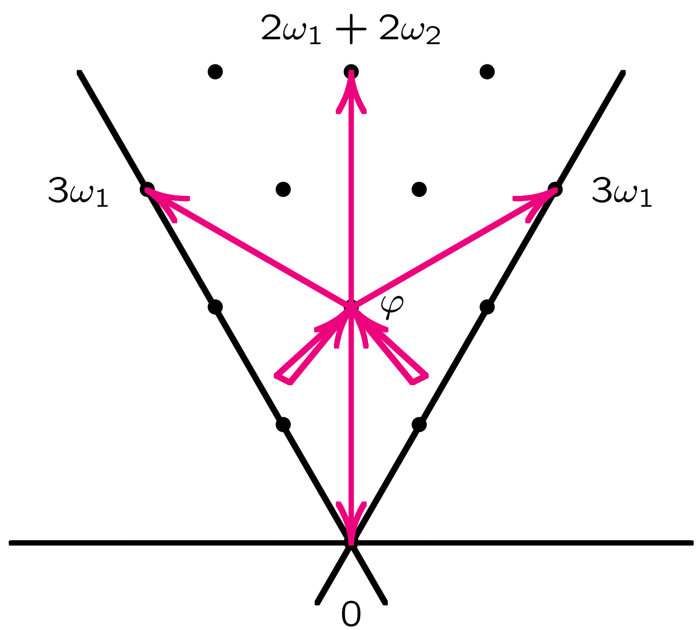
$$s_\varphi s_\varphi = s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_\varphi + s_0$$

$$s_{(2,1)} s_{(2,1)} = s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(2,2,2)}$$



$$s_\varphi s_\varphi = s_{2\omega_1+2\omega_2} + s_{3\omega_1} + s_{3\omega_2} + 2s_\varphi + s_0$$

$$s_{\yng(2,1)} s_{\yng(2,1)} = s_{\yng(4,2)} + s_{\yng(4,1,1)} + s_{\yng(3,3)} + 2s_{\yng(3,2,1)} + s_{\yng(2,2,2)}$$

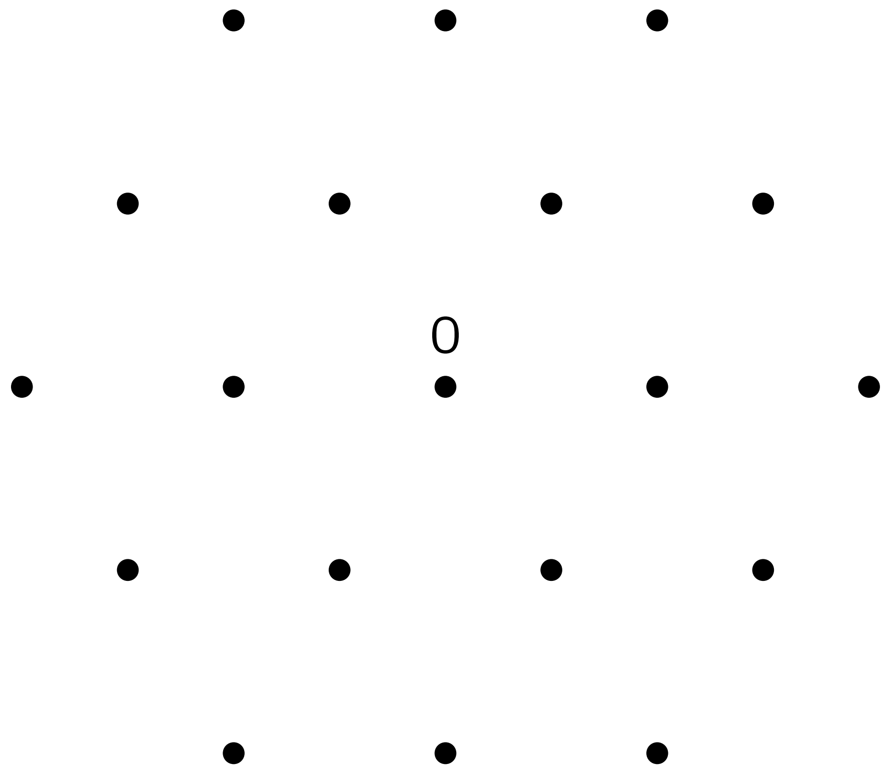


$P_\lambda(q,t)$ are the symmetric Macdonald polynomials.

$$P_\varphi(q,t) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + (2 + t + q + 2qt)\frac{1-t}{1-qt^2}x^0$$

$P_\lambda(q,q)$ are the Schur polynomials.

$$P_\varphi(q,q) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + 2$$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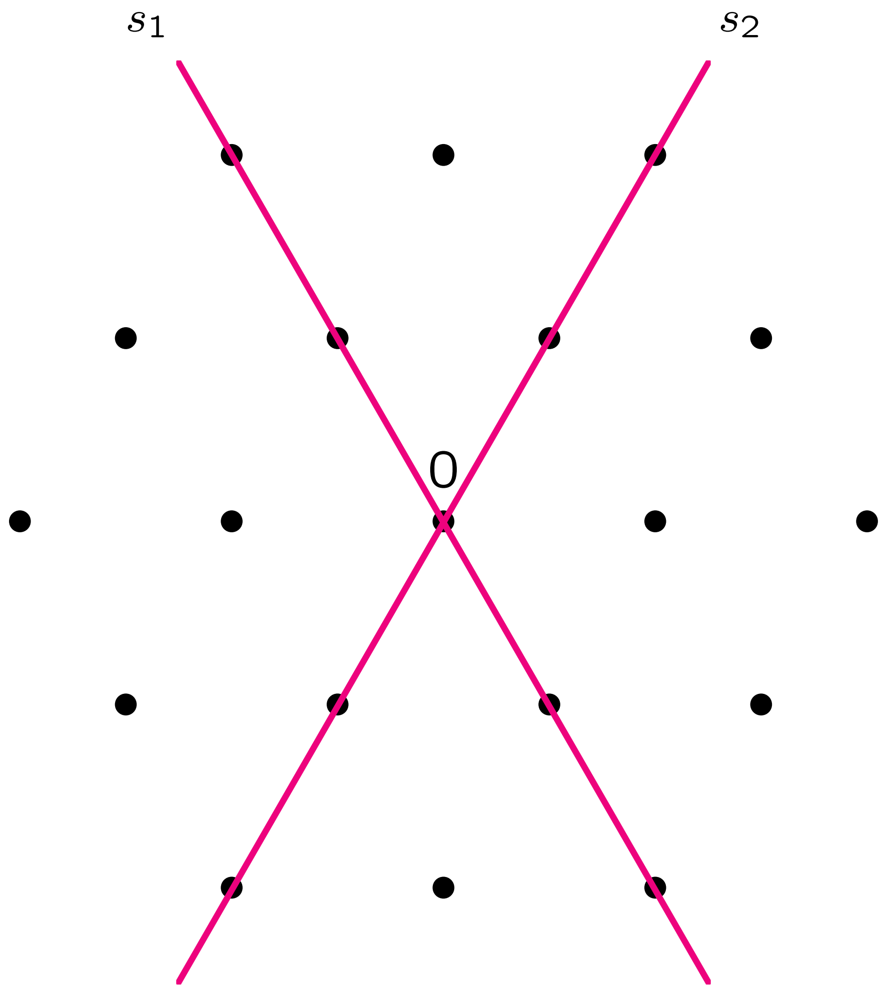

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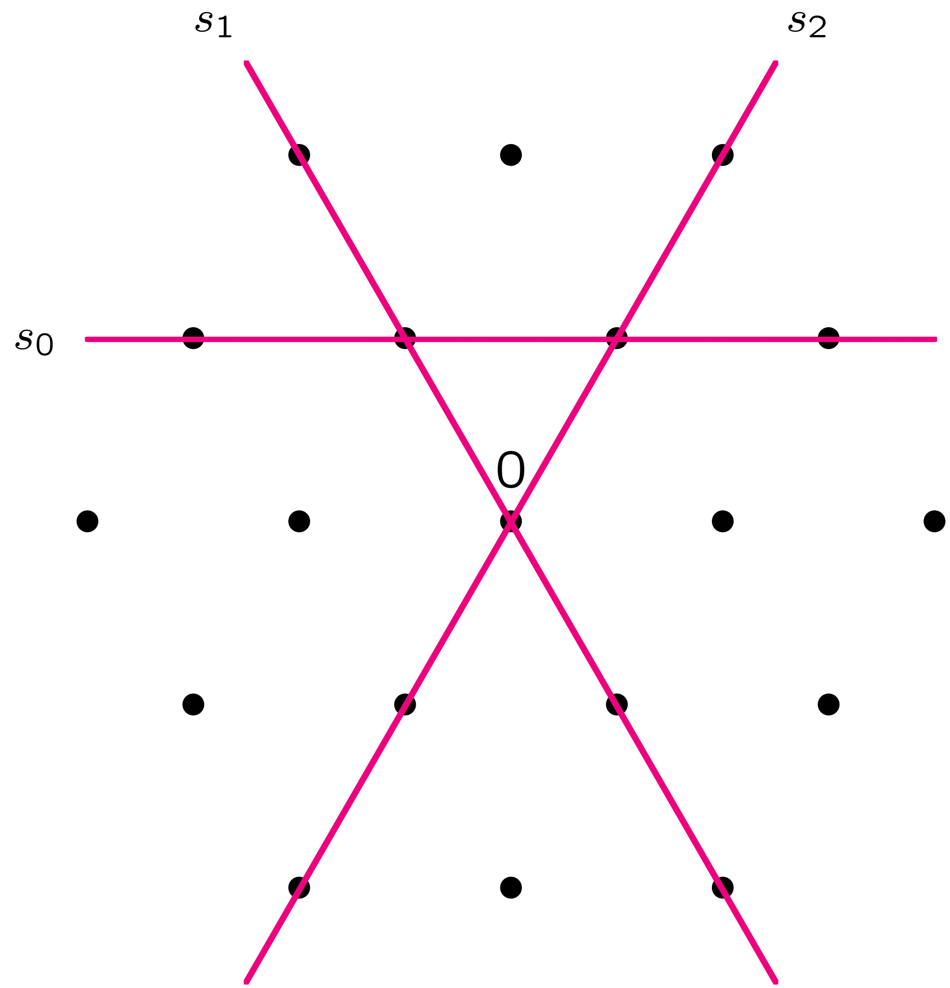
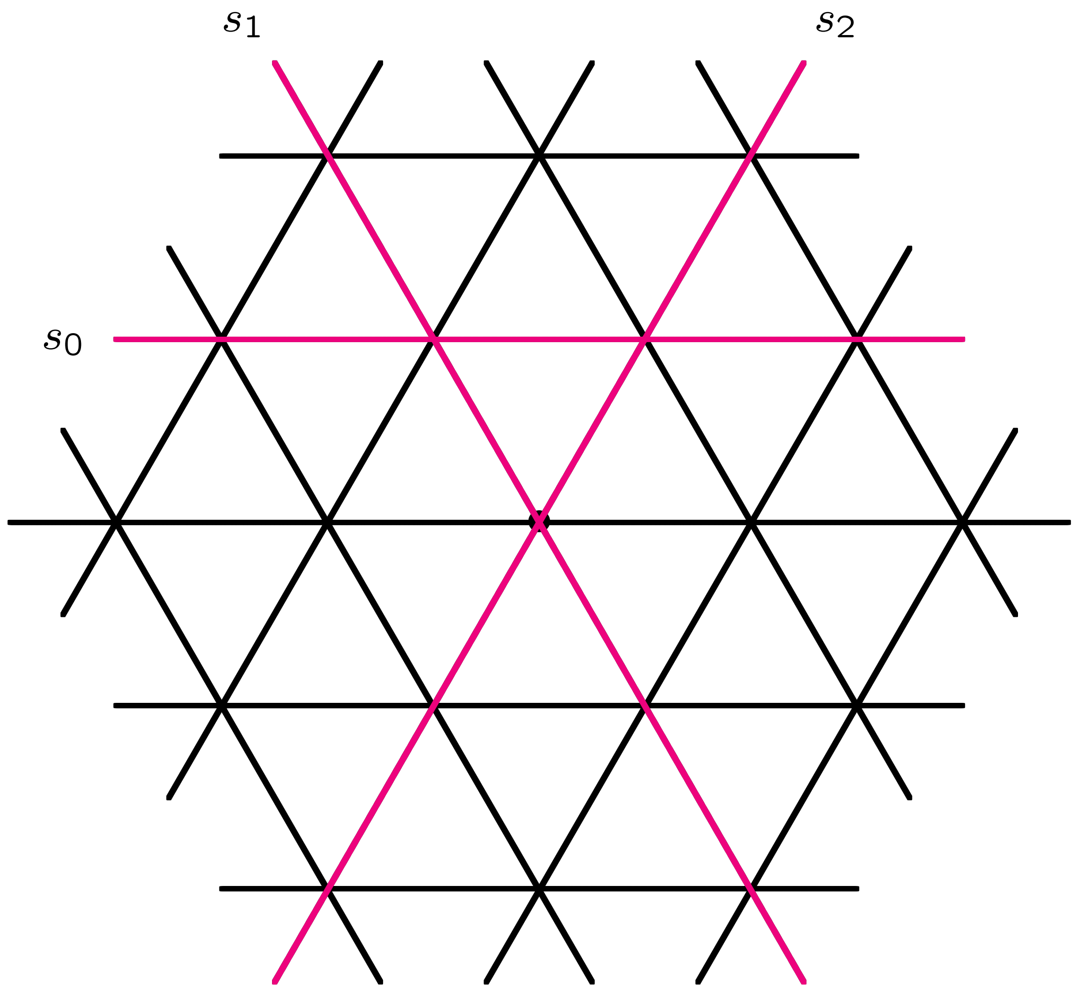
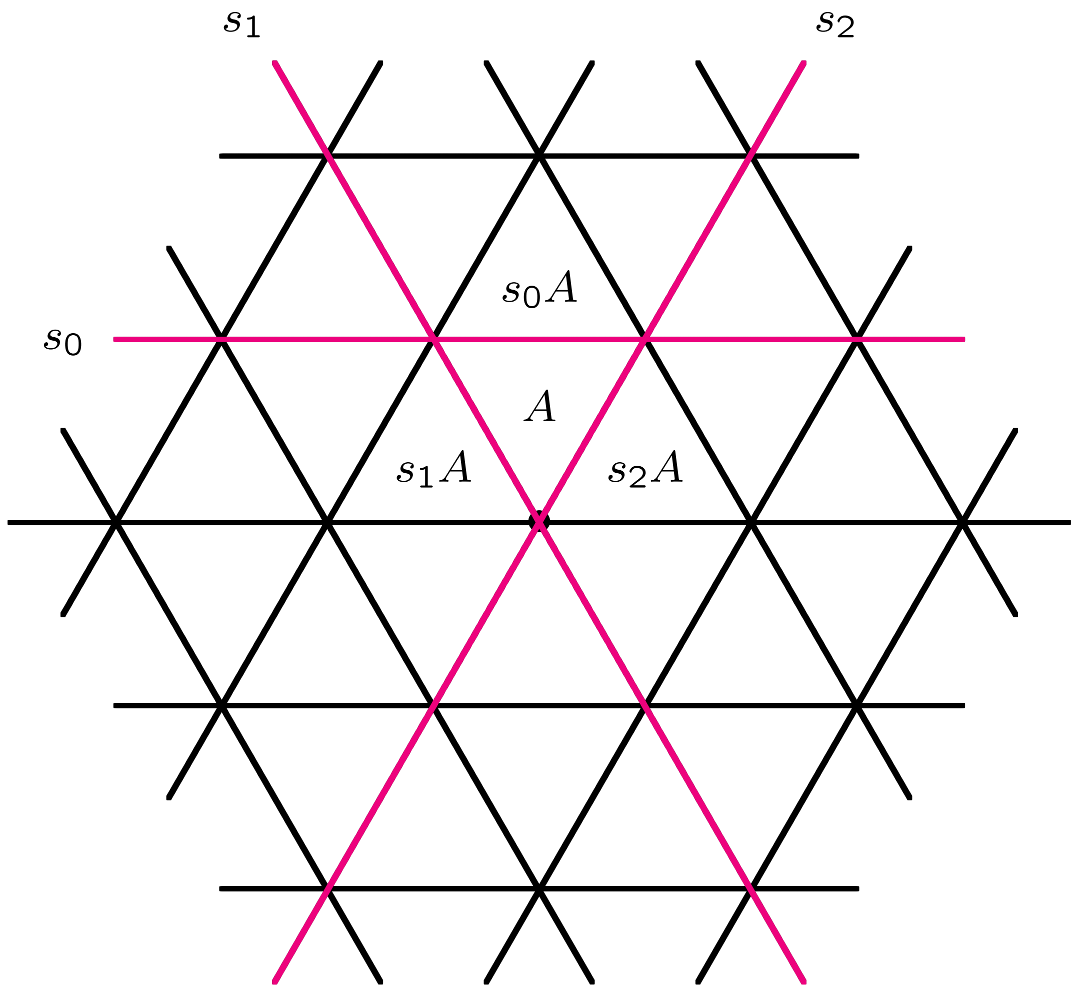
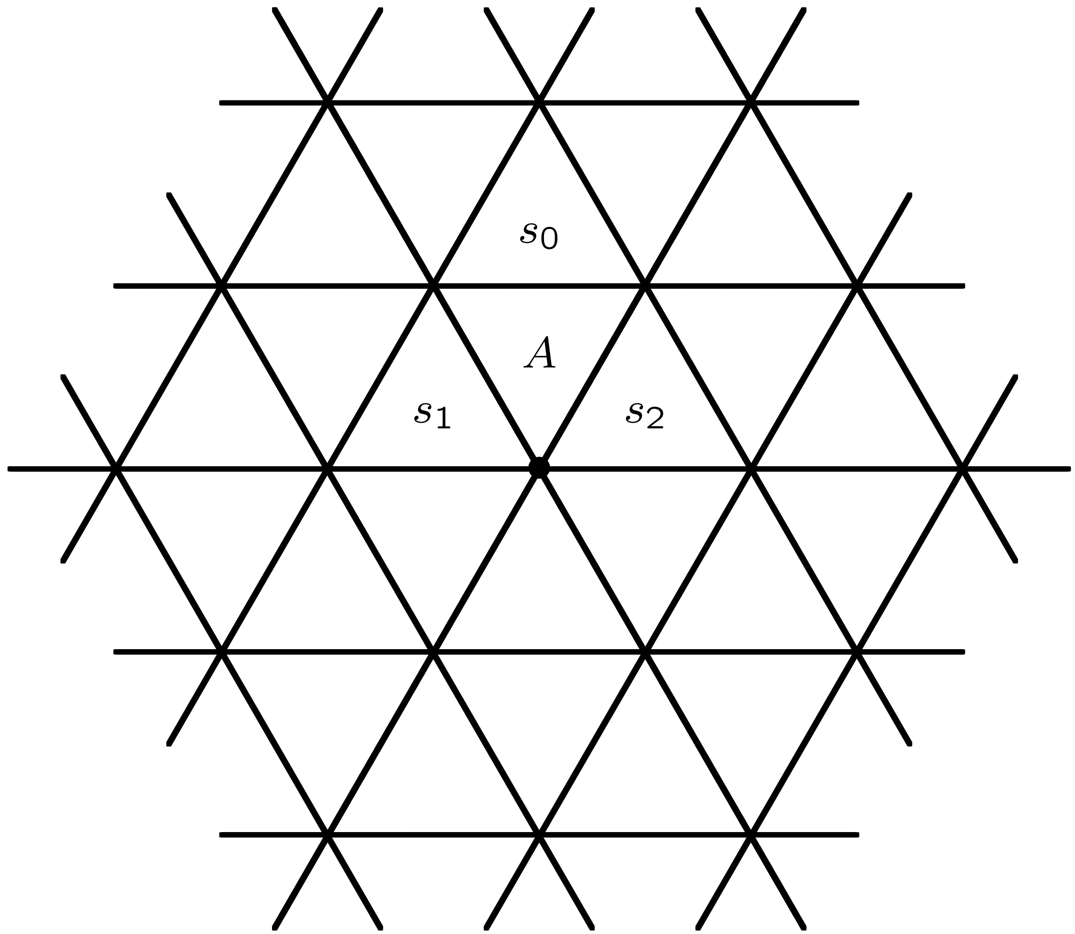


9-b

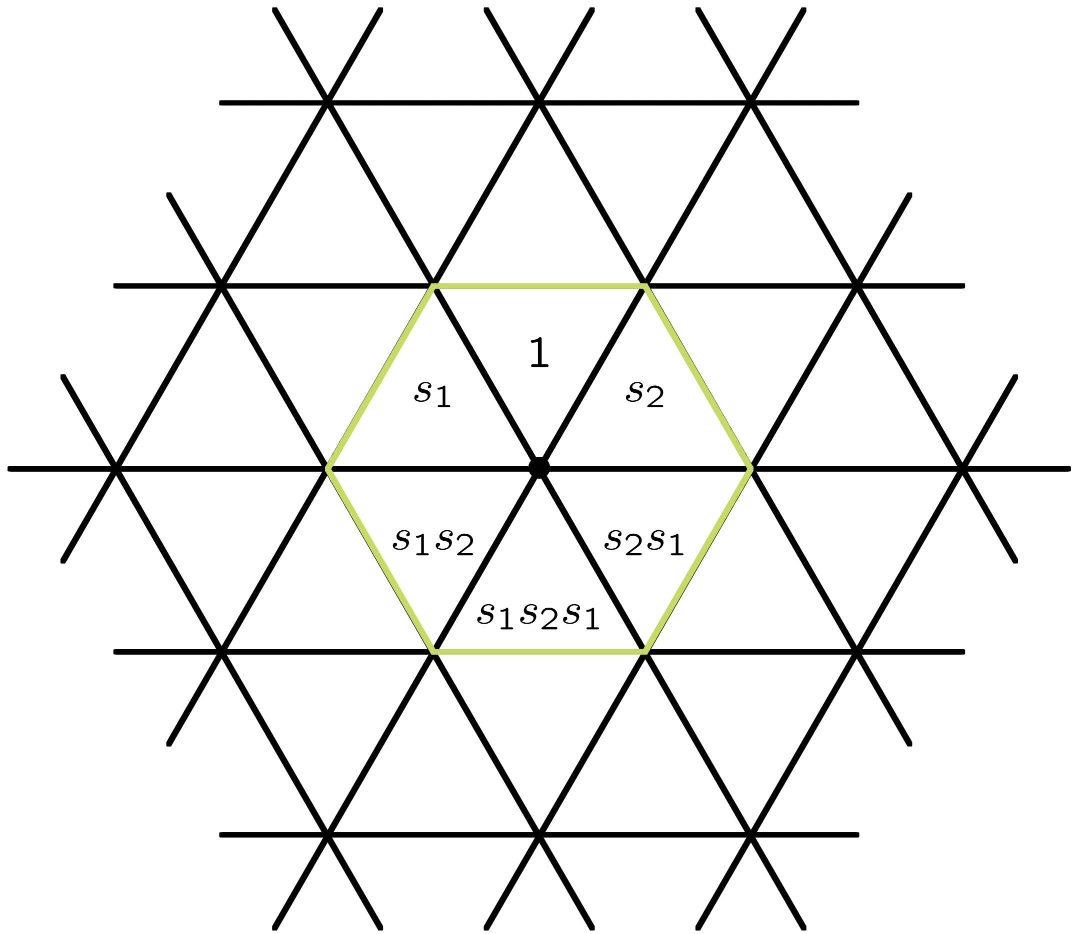

s_1
s_2
s_0



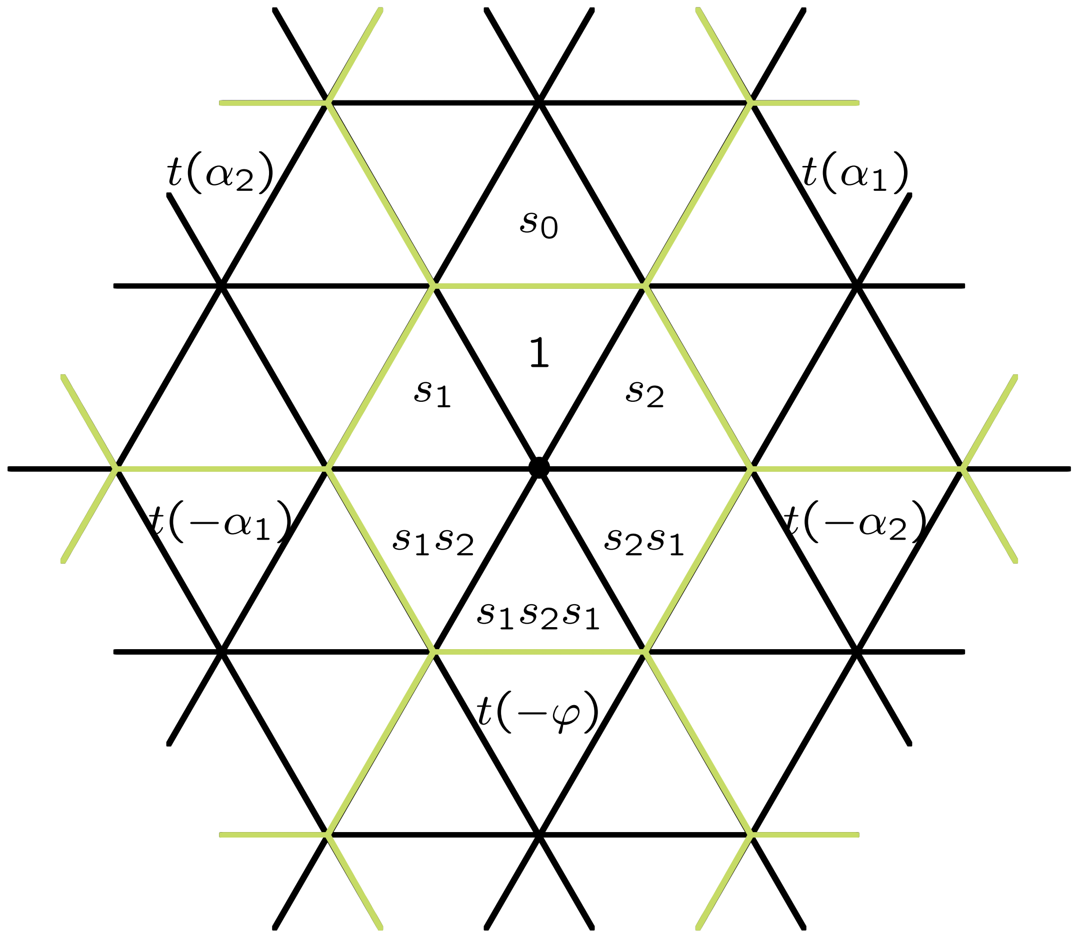
s_1
s_2
s_0
s_0A
A
s_1A
s_2A



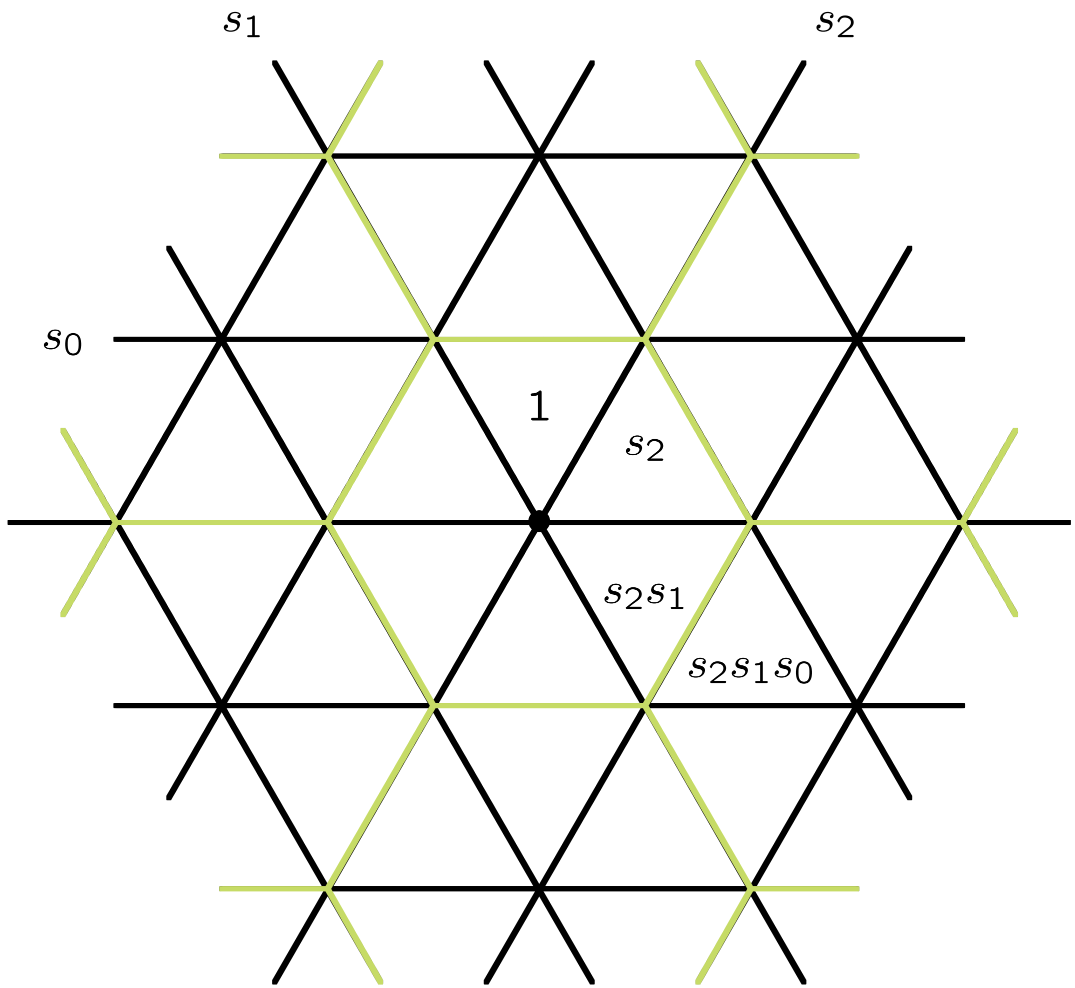
s_0
A
s_1
s_2

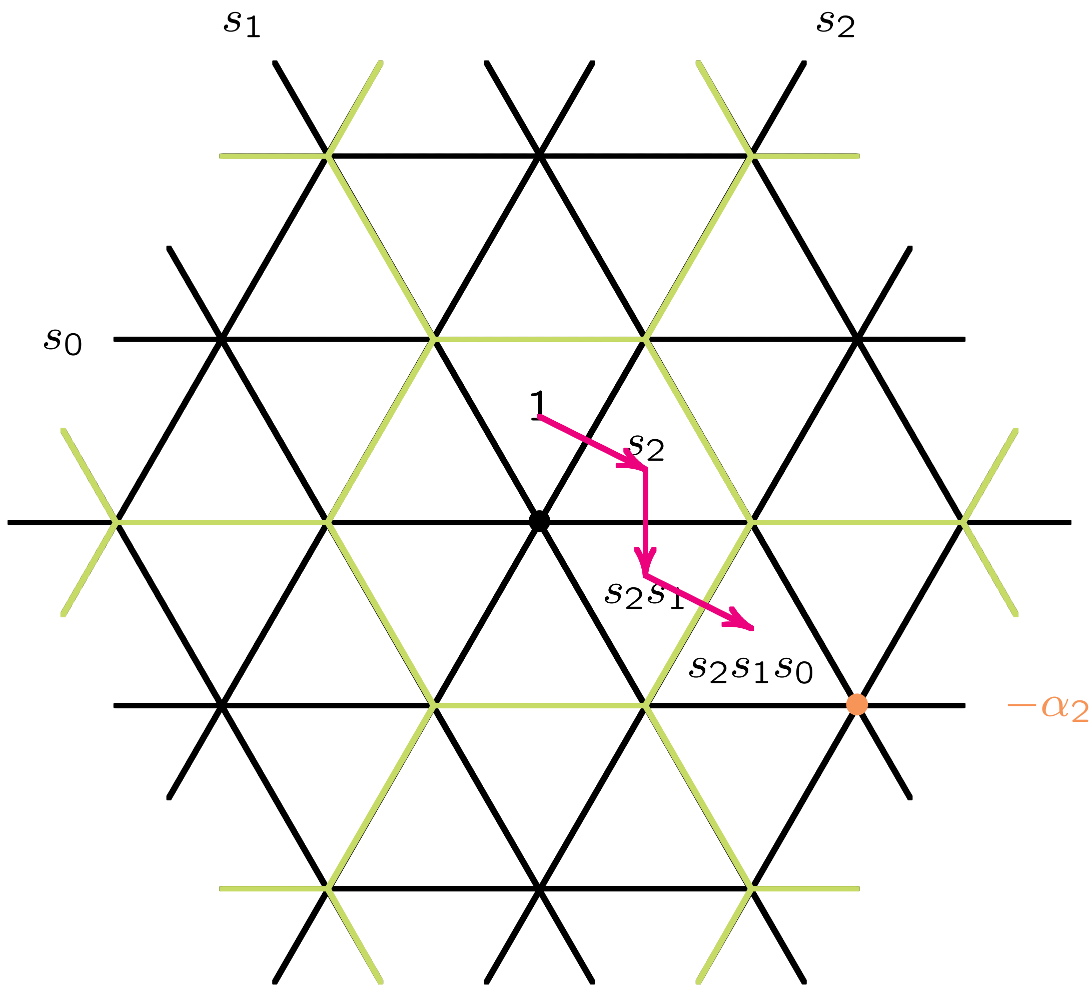


$S_3 \longleftrightarrow$ alcoves in 0-hexagon

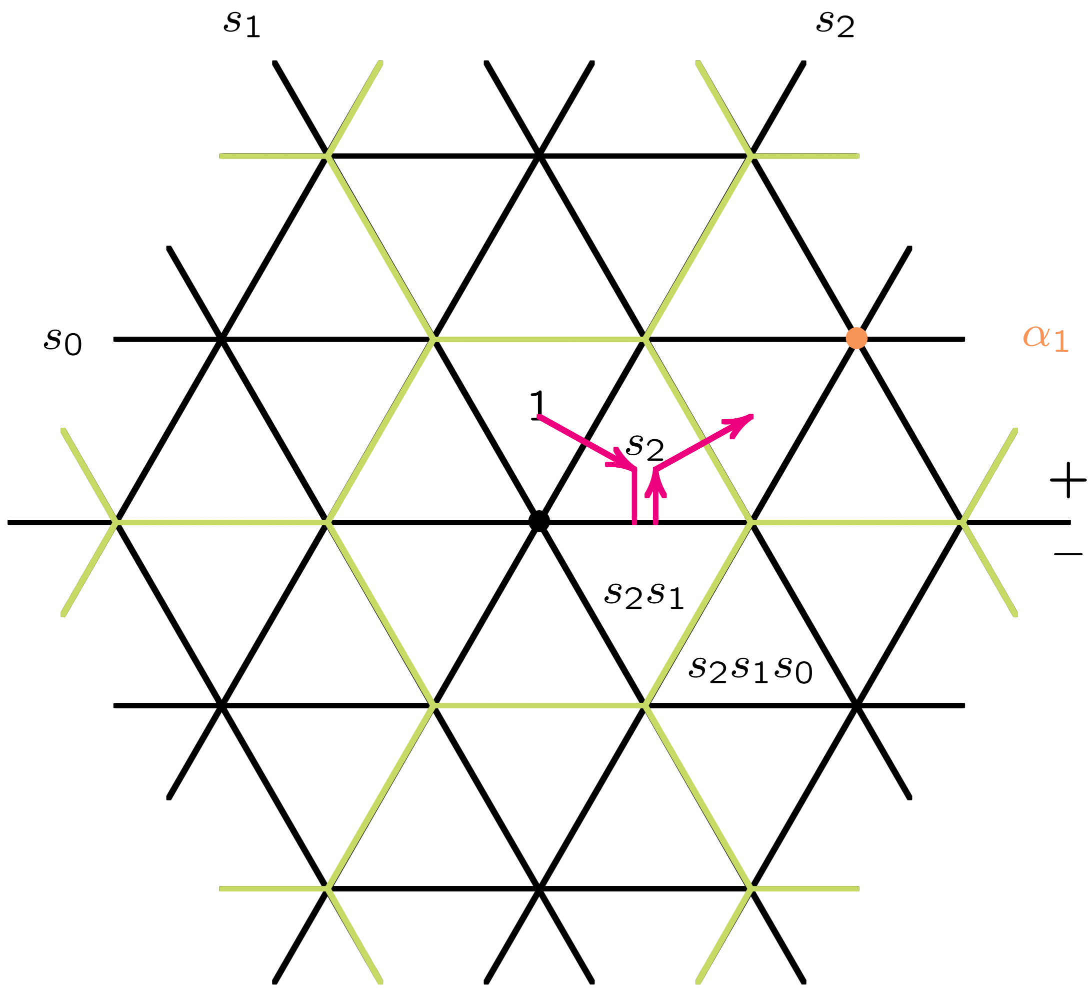


$t(\mu)W_0 \longleftrightarrow$ shifted hexagons

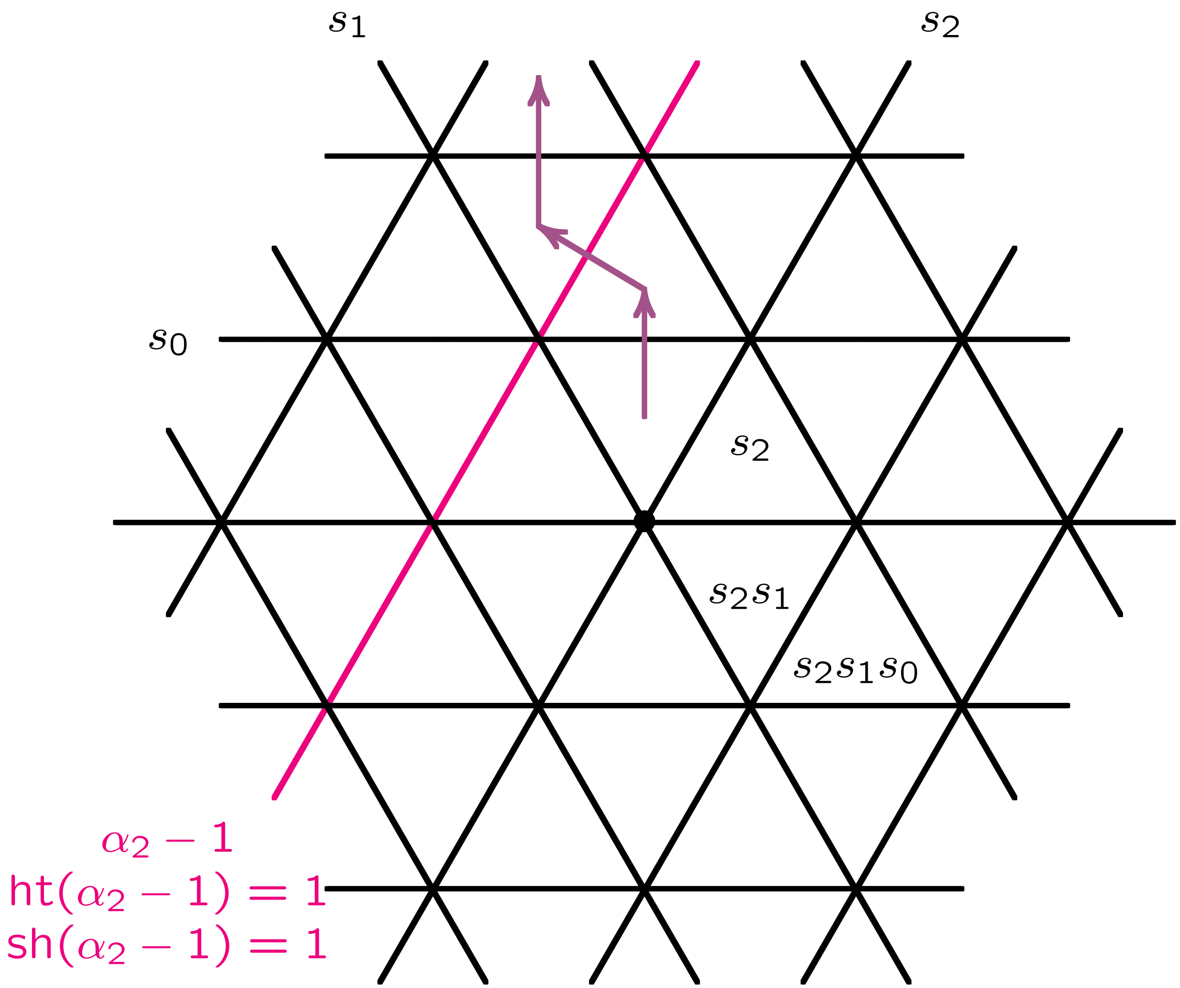

s_1
s_2
s_0
1
s_2
$s_2 s_1$
$s_2 s_1 s_0$



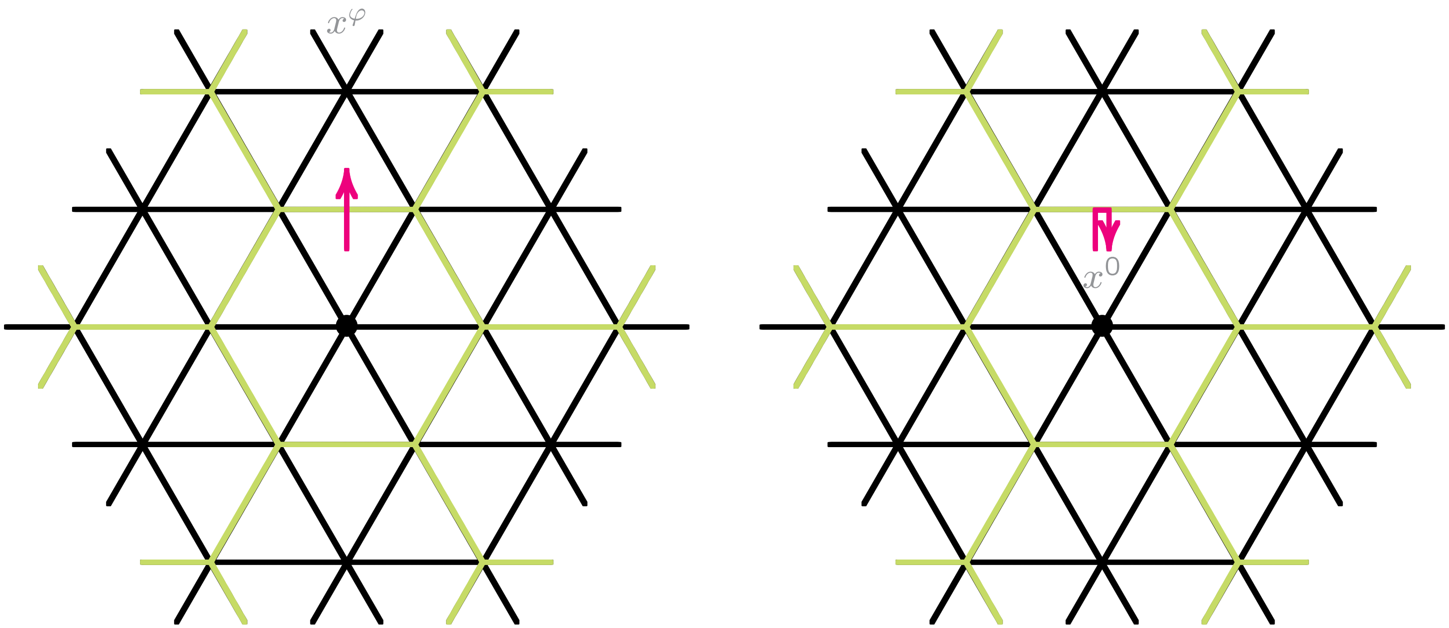
s_1
s_2
s_0
1
s_2
$s_2 s_1$
$s_2 s_1 s_0$
$-\alpha_2$



s_1
s_2
s_0
α_1
1
s_2
+
−
$s_2 s_1$
$s_2 s_1 s_0$

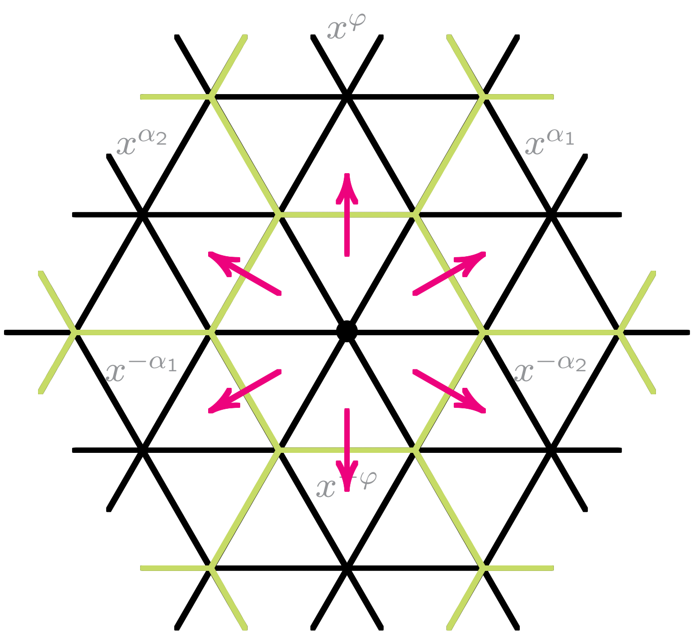

s_1
s_2
s_0
s_2
$s_2 s_1$
$s_2 s_1 s_0$
$\alpha_2 - 1$
$\mathrm{ht}(\alpha_2 - 1) = 1$
$\mathrm{sh}(\alpha_2 - 1) = 1$

$$E_\varphi(q,t) = x^\varphi + qt^2\frac{1-t}{1-qt^2}x^0$$

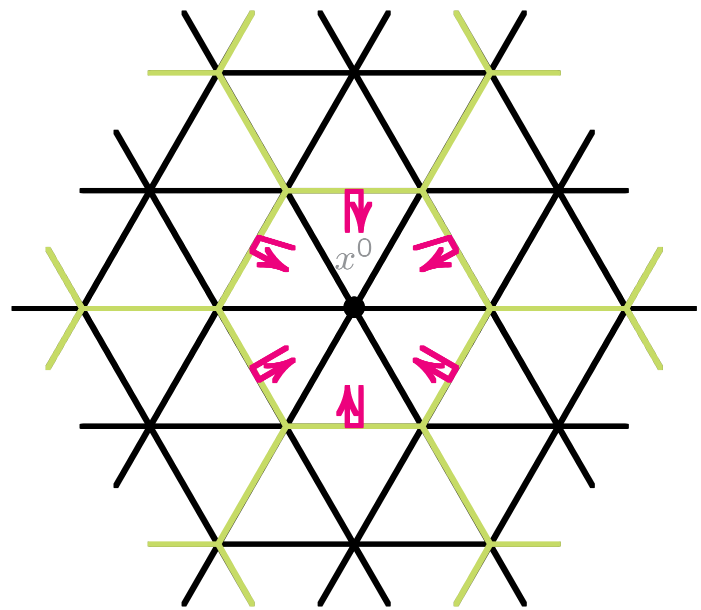


x^φ $\quad$ $qt^2\frac{1-t}{1-qt^2}x^0$

$$P_\varphi(q,t) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + (2+t+q+2qt)\frac{1-t}{1-qt^2}x^0$$

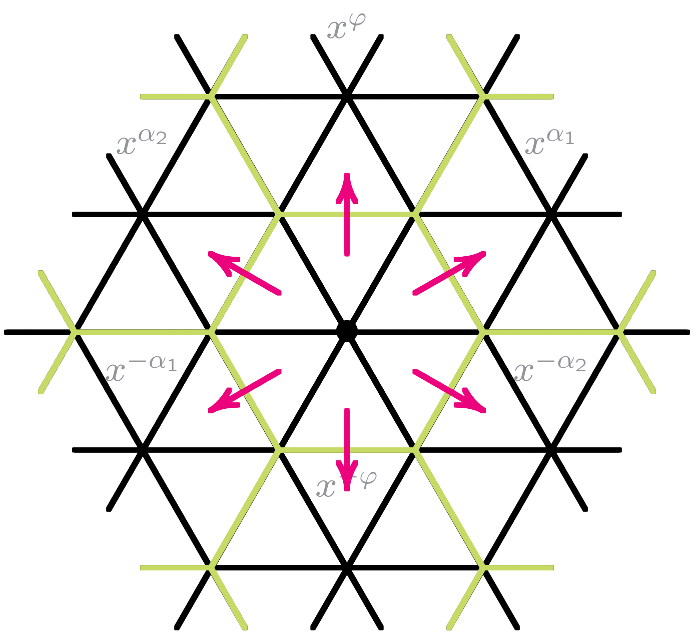


M_φ

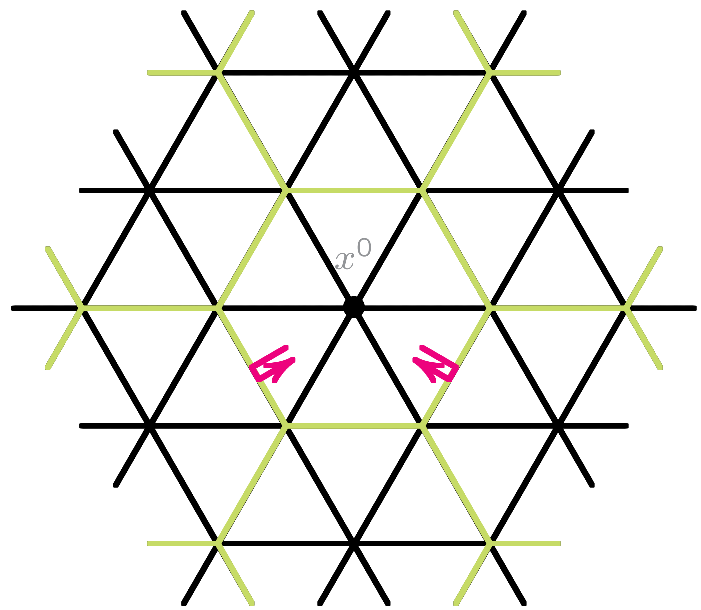


$(2+t+q+2qt)\frac{1-t}{1-qt^2}x^0$

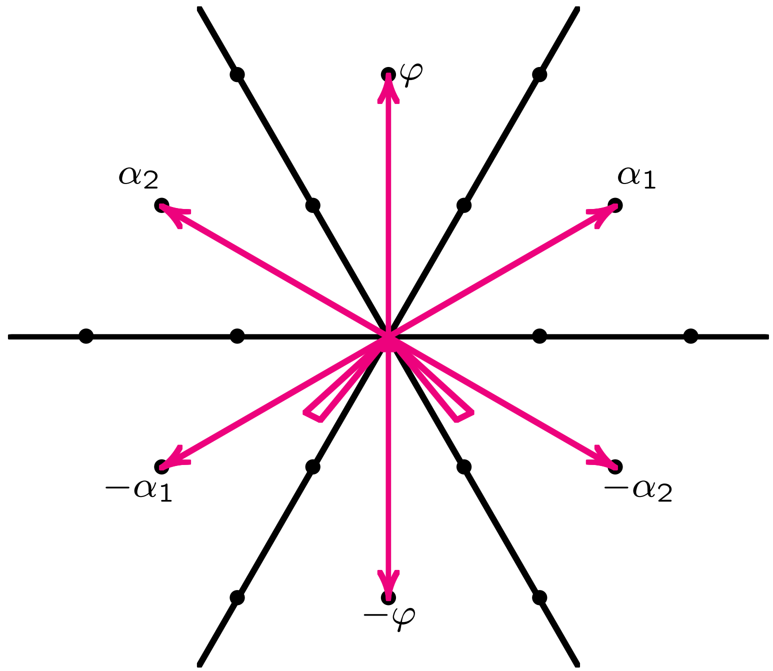
$$P_\varphi(q,q) = x^\varphi + x^{\alpha_1} + x^{\alpha_2} + x^{-\alpha_1} + x^{-\alpha_2} + x^{-\varphi} + 2x^0$$



M_φ

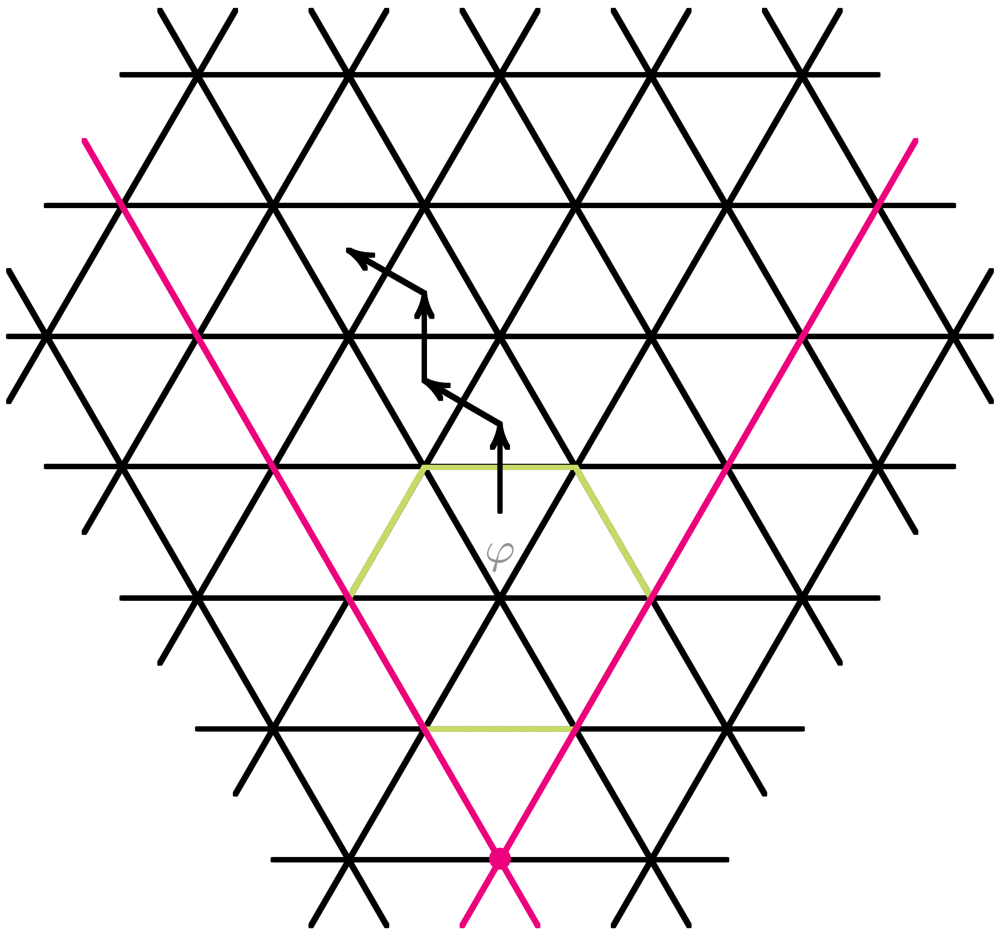


$2x^0$

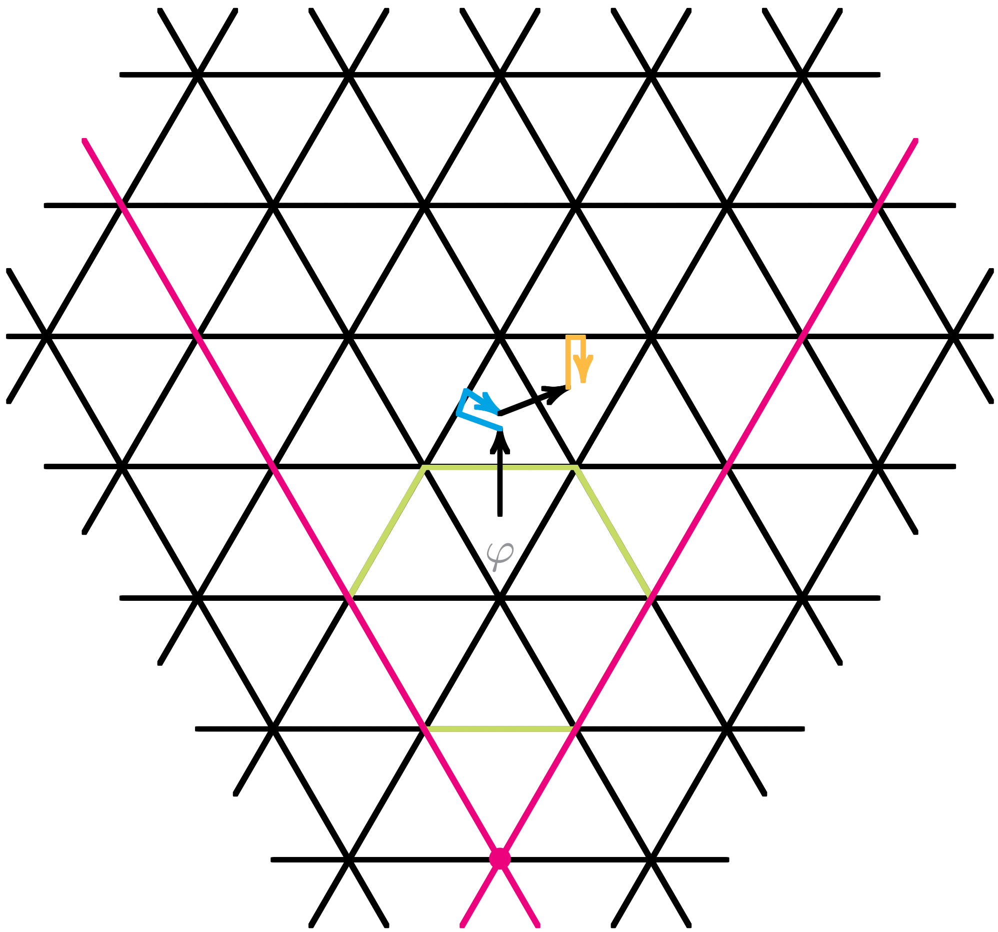


φ
α_2
α_1
$-\alpha_1$
$-\alpha_2$
$-\varphi$

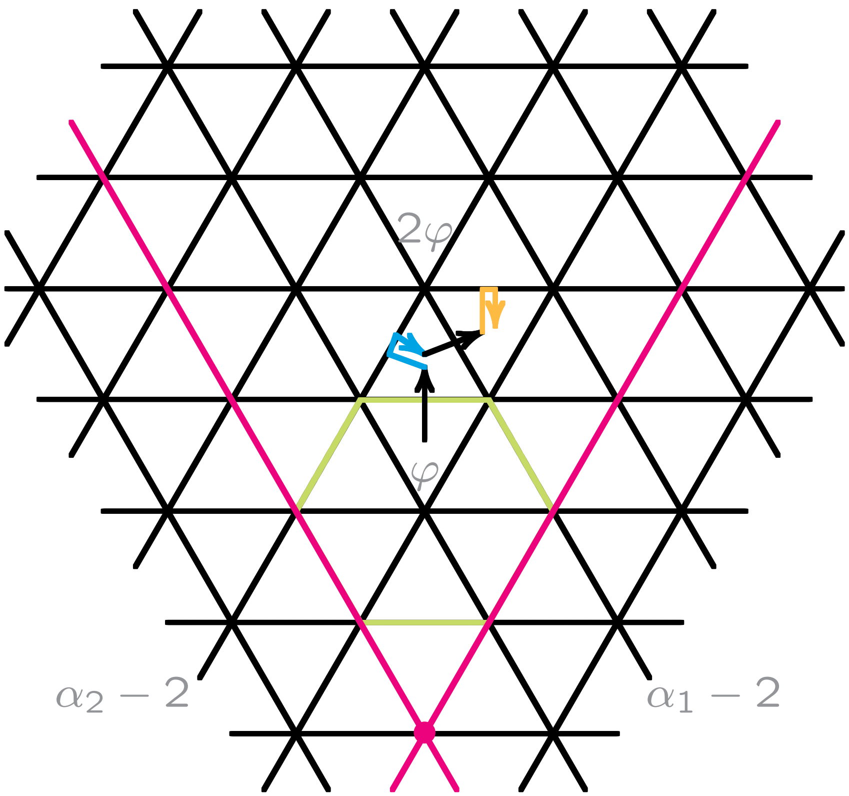
$$P_{3\omega_1}(q,t)P_\varphi(q,t)$$



$$P_{3\omega_1}(q,t)P_{\varphi}(q,t)$$



$P_{3\omega_1}(q,t)P_{\varphi}(q,t)$



$b_p = 1$

$e_p = \frac{t^{-1/2} - t^{1/2}}{1 - q^2 t}$

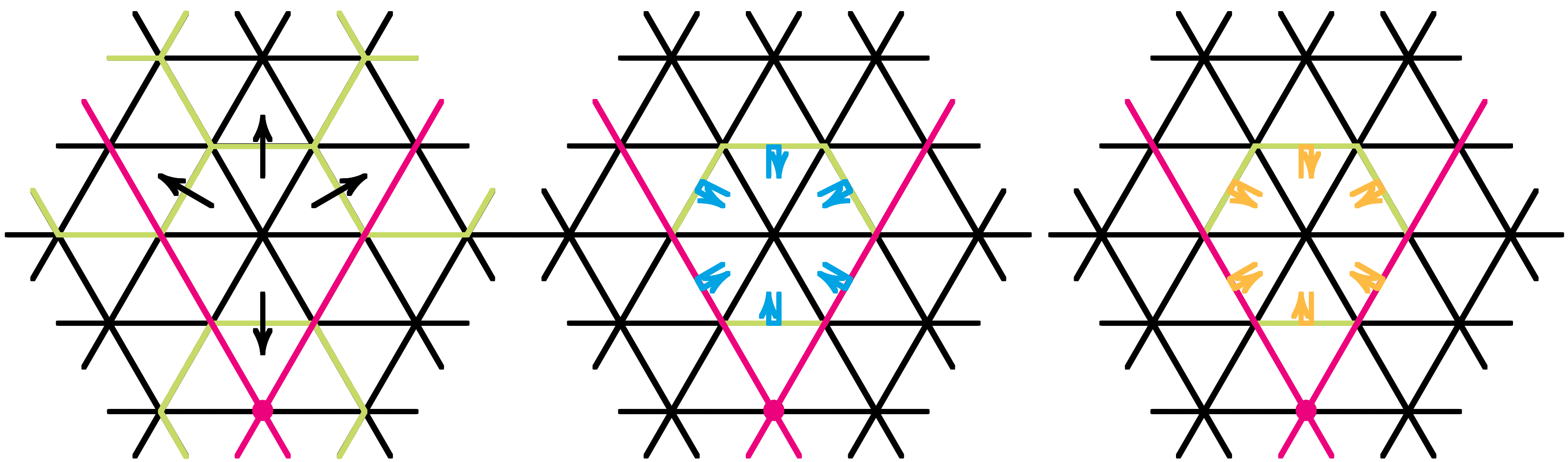
$f_p = q^2 t \frac{t^{-1/2} - t^{1/2}}{1 - q^2 t}$

$f_p = -\frac{t^{-1/2} - t^{1/2}}{1 - q^2 t}$

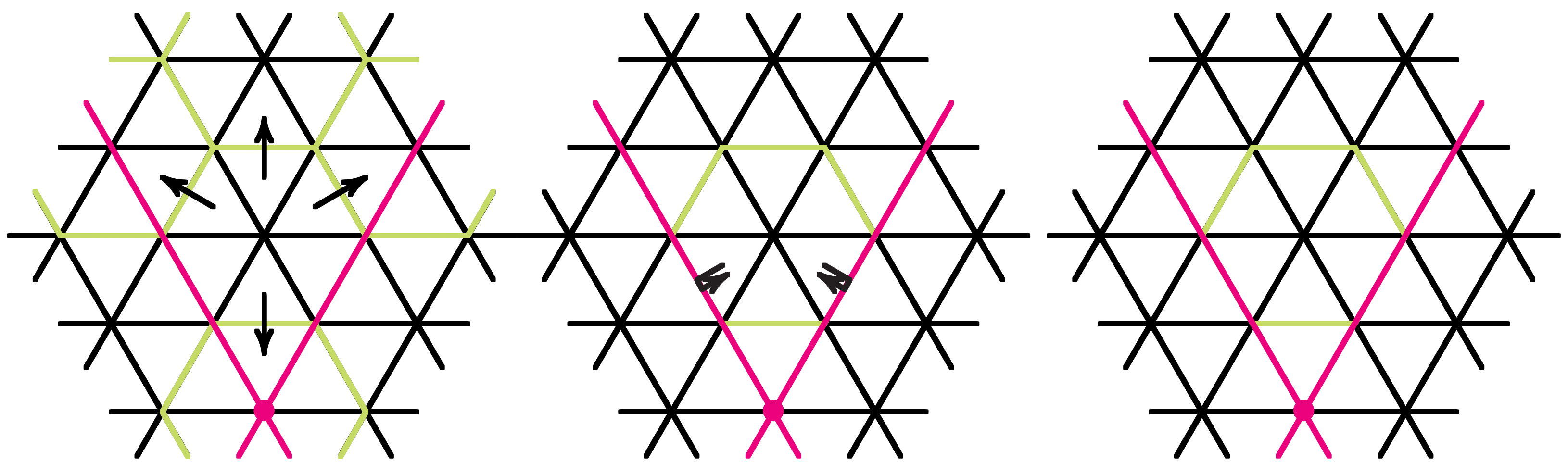
$n_p = 1$

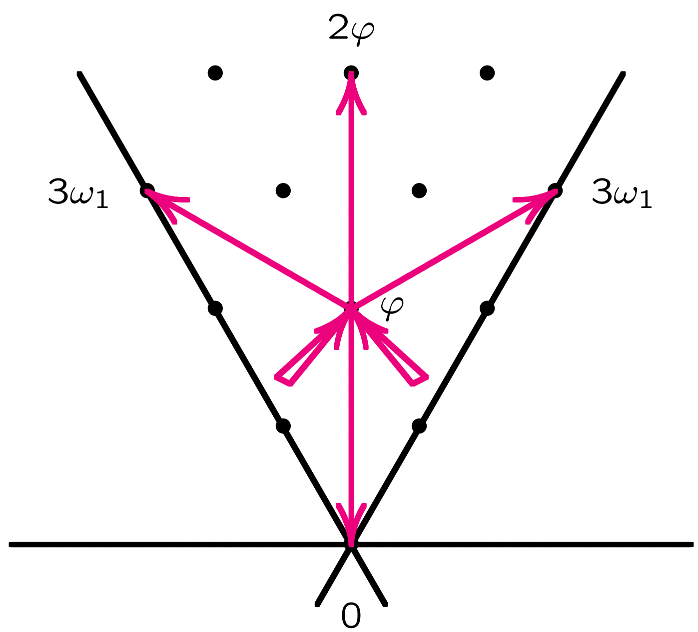
$P_{-w_0 \mathrm{wt}(p)} = P_{2\varphi}$

$$P_{\varphi}(q,t)P_{\varphi}(q,t)$$



$$P_{\varphi}(q,q)P_{\varphi}(q,q)$$




2φ
$3\omega_1$
$3\omega_1$
φ
0

$$b_p = \prod_{a^\vee \in m_\lambda^{-1}\mathcal{L}(\mathsf{i}(p))} \frac{t^{1/2} - t^{-1/2} q^{\mathsf{sh}(a^\vee)} t^{\mathsf{ht}(a^\vee)}}{1 - q^{\mathsf{sh}(a^\vee)} t^{\mathsf{ht}(a^\vee)}},$$

$$e_p = \prod_{a^\vee \in m_{-w_0 \mathsf{wt}(p)}^{-1}\mathcal{L}(\mathsf{c}(p)^{-1})} \frac{t^{-1/2} - t^{1/2} q^{\mathsf{sh}(a^\vee)} t^{\mathsf{ht}(a^\vee)}}{1 - q^{\mathsf{sh}(a^\vee)} t^{\mathsf{ht}(a^\vee)}},$$

$$f_p = \prod_{k \in \phi(p)} \frac{t^{1/2} - t^{-1/2}}{1 - q^{\mathsf{sh}(b_k^\vee)} t^{\mathsf{ht}(b_k^\vee)}} \prod_{k \in \phi^-(p)} q^{\mathsf{sh}(b_k^\vee)} t^{\mathsf{ht}(b_k^\vee)},$$

$$f_p = \prod_{k \in \phi(p)'} \frac{t^{1/2} - t^{-1/2}}{1 - q^{\mathsf{sh}(c_k^\vee)} t^{\mathsf{ht}(c_k^\vee)}} \prod_{k \in \phi(p)', r-k+1 \in \xi^+(p)} q^{\mathsf{sh}(c_k^\vee)} t^{\mathsf{ht}(c_k^\vee)},$$

$$n_p = \prod_{j \in \xi^-(p)} \frac{1 - q^{\mathsf{sh}(h_j^\vee)} t^{\mathsf{ht}(h_j^\vee)-1}}{1 - q^{\mathsf{sh}(h_j^\vee)} t^{\mathsf{ht}(h_j^\vee)}} \frac{1 - q^{\mathsf{sh}(h_j^\vee)} t^{\mathsf{ht}(h_j^\vee)+1}}{1 - q^{\mathsf{sh}(h_j^\vee)} t^{\mathsf{ht}(h_j^\vee)}}.$$