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A LITTLEWOOD-RICHARDSON RULE FOR MACDONALD POLYNOMIALS

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Abstract. Macdonald polynomials are orthogonal polynomials associated to root systems and depend on parameters q and t. The double affine Hecke algebra H is a fundamental tool for studying Macdonald polynomials, which can be constructed by applying intertwining operators on the polynomial representation of H.

Using objects known as alcove walks, we give a combinatorial description for the coefficients in the expansion of a product of two Macdonald polynomials. At q=0, the formula specializes to the formula of Schwer for Macdonald spherical functions in terms of positively folded galleries, and at q=t, this formula specializes to the formula of Littelmann for Weyl characters in terms of the Littelmann path model.

Fix a root system R reduced and irreducible. Simple roots will be denote $\alpha_1, \dots, \alpha_n$ and they span the root lattice Q. Fundamental weights $\omega_1, \dots, \omega_n$ span the weight lattice L. Denote the coroot lattice and coweight lattice by $Q^\vee$, $L^\vee$.

For $\alpha^\vee \in R^\vee$, $j \in \mathbb{Z}$, let

$$H_{-\alpha^\vee + jd} = \{x \in \mathbb{R}^n \mid \langle x, \alpha^\vee \rangle = j$$

be a hyperplane and let $s_{-\alpha^\vee + jd}$ be the reflection in $H_{-\alpha^\vee + jd}$. Also, use notation

$$s_i = s_{\alpha_i} \qquad s_0 = s_{-\varphi^\vee + d} = t(\varphi^\vee) s_\varphi.$$

The affine Weyl group of R is generated by $s_0, \dots, s_n$, and is also the semidirect product of the coroot lattice and the finite Weyl group $\cong Q^\vee \rtimes \langle s_1, \dots, s_n \rangle$. Let $W_0 = \langle s_1, \dots, s_n \rangle$. This acts on $L^\vee$ by translation and reflection. Denote translations by $t(\lambda^\vee)$.

Instead of translations by elements of the root lattice, we can work with translations by coweights. In this case, the extended affine Weyl group is

$$W^\vee = L \rtimes W_0 \cong (Q \rtimes W_0) \rtimes \Pi^\vee$$

where $\Pi^\vee \cong L/Q$ is a finite abelian group.

Fix parameters q, t. The double affine Hecke algebra $\widetilde{H}$ is generated by the group algebra of the weight lattice, an affine braid group, and Π

$$\mathbb{C}[X] = \mathbb{C}[X^{\pm\omega_1}, \dots, X^{\pm\omega_n}], T_0, \dots, T_n, \Pi, q = X^\delta.$$

We can also write

$$\widetilde{H} = \mathbb{C}[X] \otimes \langle T_1, \dots, T_n \rangle \otimes \mathbb{C}[Y]$$

where Y is isomorphic to the group algebra of the weight lattice.

Now, $\mathbb{C}[X]$ is an $\widetilde{H}$-module, where

$$T_i 1 = t^{1/2} \text{ for } 1 \le i \le n, \quad X^\mu 1 = X^\mu \text{ for } \mu \in L, \quad Y^{\lambda^\vee} 1 = t^{\mathrm{ht}(\lambda^\vee)} \text{ for } \lambda^\vee \in L^\vee.$$

This is the polynomial representation.

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Now, $\mathbb{C}[Y]$ is a family of commuting operators on $\mathbb{C}[X]$, and their eigenvectors are the nonsymmetric Macdonald polynomials

$$E_\mu(X; q, t)$$

The symmetric Macdonald polynomials are indexed by dominant weights, can be obtained from the symmetric ones by applying a symmetrizing operator

$$P_\mu(X; q, t) = \mathbf{1}_0 E_\mu(X; q, t).$$

Here, $\mathbf{1}_0$ is the unique element in $\widetilde{H}$ such that $T_i \mathbf{1}_0 = t^{1/2} \mathbf{1}_0$ for $1 \leq i \leq n$. The symmetric polynomials form a basis for the subspace of W_0-invariant polynomials $\mathbb{C}[X]^{W_0}$.

Each E_μ is a product of Y-intertwining operators

$$\tau_i = T_i + \frac{t^{-1/2} - t^{1/2}}{1 - Y^{-\alpha_i}}, \qquad \text{for } 1 \leq i \leq n,$$
$$\tau_0^\vee = (X^\varphi T_{s_\varphi})^{-1} + \frac{t^{-1/2} - t^{1/2}}{1 - qY^\varphi}.$$

where if $m_\mu = \pi s_{i_1} \cdots s_{i_r}$ is a reduced expression of the minimal length coset representative m_μ of $t(\mu) W_0$, then $E_\mu = \pi^\vee \tau_{i_1}^\vee \cdots \tau_{i_r}^\vee mathbf1$.

We can compute these combinatorially by using alcove walks.

$$X^\mu E_\lambda \mathbf{1} = \sum_p f_p X^{\mathtt{wt}(p)} \mathbf{1}$$
$$= \sum_p n_p f_p' E_{\mathtt{e}(p)^{-1}} \mathbf{1},$$

where the first sum is over walks of type m_λ starting in $t(\mu)$, and the second sum is over walks of type $x^{-\mu}$ starting in w^{-1}, contained in the dominant chamber C.

Smashing these two together, we get

Theorem 0.1.

$$P_\mu P_\lambda = \sum_p b_p e_p n_p f_p f_p' P_{-w_0 \mathtt{wt}(p)},$$

is a sum over paths of type m_μ^{-1} starting in the alcoves of the coset $t(-w_0\lambda) W_0$, and is contained in the dominant chamber, and whose folds are 2-coloured.

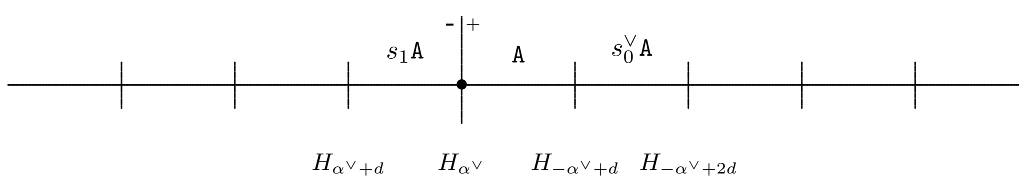
Alcoves are connected components of

$$\mathbb{R}^n - \{H_{-\alpha^\vee + jd} \mid \alpha \in R, j \in \mathbb{Z}\}.$$

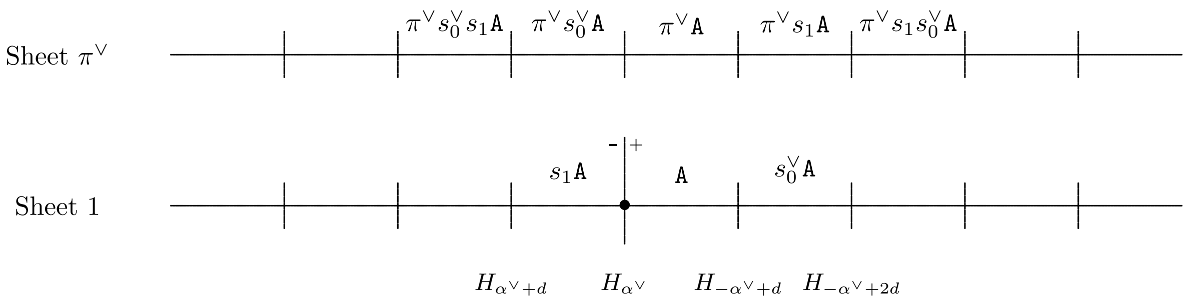
The affine Weyl group $Q^\vee \rtimes W_0$ acts simply transitively on it, so there is a 1-1 correspondence between alcoves and elements of the group.

Example 0.2. A_1. $W_0 = \{1, s_1\}$.

Sheet 1



In this root system, the root lattice is $Q = \mathbb{Z}\alpha$ and the weight lattice is $L = \mathbb{Z}\omega = \frac{\mathbb{Z}}{\alpha}$, so $\Pi^\vee = L/Q = \mathbb{Z}/2\mathbb{Z}$. To work with the extended group, we need to add copies of alcoves. In this case, need 2 copies total. Let $\Pi^\vee = \{1, \pi^\vee\}$. The picture is



We can but a numbering on these hyperplane walls. Given $w \in W^\vee$ with a reduced expression $w = \pi^\vee s_{i_1} \cdots s_{i_r}$, an alcove walk of type w is a sequence of steps where the kth step is a step across, (or a fold against) a wall with number i_k.

The coefficients b_p and e_p keep track of where the walk begins and ends, f_p, f'_p are two different ways to read fold statistics, n_p keeps track of negative crossings. Each of these is a product of rational functions in q and t.

Example 0.3. Compute $P_{3\omega} P_{k\omega}$. There are 18 paths in all.

Remark 0.4. In Type A, if ω_r is a fundamental weight, then $P_{\omega_r} = e_r$ is the rth elementary symmetric function. The path formula says that $e_r P_\lambda$ is a sum over $(n+1)!$ paths, each is a "change in sheets" so there are no folds and no negative crossings. So

$$e_r P_\lambda = \sum_p b_p e_p P_{-w_0 \mathtt{wt}(p)}.$$

Macdonald has a tableaux formula

$$e_r P_\lambda = \sum_\kappa \prod_s \frac{1 - q^{a_\kappa} t^{l_\kappa + 1}}{1 - q^{a_\kappa} t^{l_\lambda + 1}} \frac{1 - q^{a_\lambda + 1} t^{l_\lambda}}{1 - q^{a_\kappa + 1} t^{l_\lambda}}$$

over partitions κ which can be obtained from λ by adding r boxes, with at most one per row, and the product is over boxes s such that a box has been added in that column, but no box has been added in that row. The q, t coefficients are arm length and leg lengths.

Assuming λ has no equal parts, then there are $\binom{n+1}{r}$ such partitions κ.

Observe that the stabilizer of ω_r is $W_{\omega_r} = S_r \times S_{n+1-r}$. Using this, we can achieve a "compression" of the path formula into $\binom{n+1}{r}$ paths, so that each path corresponds to one tableaux in Macdonald's formula. This is a first step towards compressing the product formula.