Critical Points

1. For the following functions, determine whether there exist maximum and minimum in the given region.

   (a) $f(x, y) = e^{\cos(x^2)}(y^3 + xy + 3)$, $x^2 + y^2 \leq 2$

   (b) $f(x) = \sin(x)\sin(x^2)$, $x \in (-\infty, \infty)$

   (c) $f(x, y) = \tan(x^2 + y^2)$, $x \in [0, \sqrt{\pi}/2)$ and $y \in [0, \sqrt{\pi}/2)$

2. Find all critical points in the domains of following functions. Which of the critical points are local minima or local maxima?

   (a) $f(x, y) = xy$

   (b) $f(x, y, z) = x^2 y$

(c) $f(x, y, z) = e^{x+y} z$

(d) $f(x, y, z) = \sin(x + 2y)$

(e) $f(x, y) = \sin(x) \cos(y)$

(f) $f(x, y) = (1 - x^2 - y^2)^2$

(g) $f(x, y) = x^2 y - \frac{1}{3} y^3$

3. $f(x, y) = x^2 + 2xy + y^2 + ax + by$. If $a = b$, find all critical points. If $a \neq b$, prove that there are no critical points.

Solutions:

1. (a) Yes

   (b) We don't know

   (c) There is minimum but no maximum.

2. (a) $(0,0)$: saddle.

   (b) Critical points: $(0,y)$ for all y. If $y>0$, it's a local maximum. If $y<0$, it's a local minimum.

   (c) No critical point.

   (d) Critical points: (x,y) satisfying $x+2y=(k+\frac{1}{2})\pi$, where k is any integer. If k is even, it's a local maximum; otherwise, it's a local minimum.

   (e) Critical points: $((m+\frac{1}{2})\pi, n\pi)$ and $(m\pi,(n+\frac{1}{2})\pi)$, where m and n are integers.

   Maxima: $((m+\frac{1}{2})\pi, n\pi)$ when m and n are both even, or both odd.

   Minima: $((m+\frac{1}{2})\pi, n\pi)$ when only one of m and n is odd.

   Others are saddle points.

   (f) $(0,0)$: local maximum. (x,y) satisfying $x^2+y^2=1$: local minima.

   (g) $(0,0)$: neither maximum nor minimum.

3. If $a=b$, then critical points are on the line $2x+2y+a=0$. If $a\neq b$, no critical points, since lines $2x+2y+a=0$ and $2x+2y+b=0$ have no intersections.