Green's Theorem

1. Apply Green's Theorem to evaluate the following integrals:

   (a) $\oint_C (y^2\ dx + x^2\ dy)$, where C is the counter clockwise traversed boundary of the trangle bounded by $x = 0$, $x + y = 1$ and $y = 0$.

   (b) $\oint_C (6y + x)\ dx + (y + 2x)\ dy$, where C is the counter clockwise traversed boundary of the circle $(x - 2)^2 + (y - 3)^2 = 4$.

   (c) $\oint_C (2x + y^2)\ dx + (2xy + 3y)\ dy$, where C is the counter clockwise traversed boundary of any simply connected region.

2. Compute the following flux integrals:

   (a) of the field $\vec{F} = (\arctan \frac{y}{x})\vec{i} + \ln(x^2 + y^2)\vec{j}$ outward the curve C, which is the counter clockwise traversed boundary of the region defined by the polar coordinate inequalities $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.

(b) of the field $\vec{F} = xy\vec{i} + y^2\vec{j}$ outward the curve C, which is the counter clockwise traversed boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

3. **Calculating Area with Green's Theorem**

   (a) Apply Green's Theorem to show that if a curve C is the counter clockwise traversed boundary of a simply connected region R, then we have

   $$\text{Area of } R = \frac{1}{2}\oint_C x\ dy - y\ dx$$

   (b) Use the formula above to compute the area of the ellipse bounded by $\vec{x}(t) = (a\cos t)\vec{i} + (b\sin t)\vec{j}$, $0 \le t \le 2\pi$.

   (c) Compute the area of the region bounded by $\vec{x}(t) = (\cos^3 t)\vec{i} + (\sin^3 t)\vec{j}$, $0 \le t \le 2\pi$.

4. Give 3 equivalent statements of the following one in a 2-dimensional space:

   (a) The vector field $\vec{F}(x, y)$ is conservative.

   (b)

   (c)

   (d)

Solutions:

1. (a) 0

   (b) -16π

   (c) 0

2. (a) 2

   (b) $\frac{1}{5}$

3. (a)

   (b) $ab\pi$

   (c) $\frac{3\pi}{8}$

4. Assume the domain of $\vec{F} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$ is simply connected. Then the following are equivalent:

   (a) The vector field $\vec{F}(x,y)$ is conservative.

   (b) $\oint \vec{F} \cdot d\vec{x} = 0$ for all curve C in the domain of $\vec{F}$.

   (c) There exists a function $f(x,y)$ such that $f_x(x,y) = P(x,y)$ and $f_y(x,y) = Q(x,y)$.

   (d) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.