Implicit Functions and Chain Rule of more Variables

1. Let $f(x, y) = \ln(2 + 2x + e^y)$. Use implicit functions theorem to show that the level set C of f through the point $(1, 0)$ happens to be the graph of a function $y = g(x)$, and compute $g'(1)$. (In your homework, you will see another way to find $g'(1)$.)

2. (a) Consider the level set of $f(x, y) = C$ passing through a point (x_0, y_0). If the gradient at this point is zero vector, i.e. $\vec{\nabla} f(x_0, y_0) = \vec{0}$, can we use implicit function at this point?

   (b) Recall some examples you have seen that satisfy all the conditions in part (a).

   (c) Problem 11 on page 73.

3. Show that the functions $x^2 - y^2 = 1$ and $xy = 1/2$ represent the same function under some coordinate transformation.

4. Consider a curve $(u(t), v(t))$. Show that the coordinate transformation $x(u, v) = u \cos\theta + v \sin\theta$, $y(u, v) = -u \sin\theta + v \cos\theta$ preserves the arc length.

5. For some function f, we are told that at the point with Cartesian coordinates $(4, 3)$, one has

$$f_r = 3, \quad f_\theta = 6.$$

Compute the gradient $\vec{\nabla} f(4, 3)$.

Solutions:

1. $g'(1) = 2$
2. (a) No.

   (b) $f(x, y) = x^2 - y^2 + y^3$.

   (c)
3. Use $x = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v$, $y = -\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v$.
4.

$$\begin{aligned} & \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt \\ = & \int_a^b \sqrt{(x_u u_t + x_v v_t)^2 + (y_u u_t + y_v v_t)^2} dt \\ = & \int_a^b \sqrt{(\cos\theta u'(t) + \sin\theta v'(t))^2 + (-\sin\theta u'(t) + \cos\theta v'(t))^2} dt \\ = & \int_a^b \sqrt{u'(t)^2 + v'(t)^2} dt. \end{aligned}$$

5. $\begin{pmatrix} 42/25 \\ 69/25 \end{pmatrix}$