Lagrange Multipliers Method

1. Minimize $f(x, y, z) = x + 3y + 5z$ subject to $xyz = 1$ and x, y, z all positive.

2. Consumers can buy apple juice, orange juice, and pear juice at a cost of \$2 per gallon of apple juice, \$3 per gallon of orange juice, and \$6 per gallon of pear juice. If the consumers buy x, y, z gallons of apple juice, orange juice, and pear juice, then they derive a "utility" $U(x, y, z) = xyz$ from this purchase.

   How much apple, orange, and pear juice will a consumer buy if they want to maximize their utility function $U(x, y, z)$, and if they will spend exactly \$90?

   Use the method of Lagrange multipliers to solve this problem.

3. Use the Lagrange multiplier method to find the points on the ellipse $2x^2 + y^2 = 8$ nearest to and farthest from the point $(0, 1)$. Clearly state the function whose extrema you wish to find and the constraint of the problem.

   Note: You may use the facts that $\sqrt{2} \approx 1.4$ and $\sqrt{3} \approx 1.7$.

Solutions:

1. At $(\sqrt[3]{15}, \sqrt[3]{15}/3, \sqrt[3]{15}/5)$, f reaches its minimum $3\sqrt[3]{15}$.
2. Constraint set is $g(x, y, z) = 2x + 3y + 6z = 90$. Maximum: $U(15, 10, 5) = 750$.
3. Consider function $f(x, y) = x^2 + (y - 1)^2$, which is the square of the distance to the point $(0, 1)$. (This is much simpler than setting f to be the distance). So we want to find the maximum and minimum of f on the constaint set $g = 2x^2 + y^2 = 8$. By using Lagrange multiplier method, the maximum is $f(0, -2\sqrt{2}) = 9 + 2\sqrt{2}$ and the minimum is $f(\sqrt{2}, 2) = f(-\sqrt{2}, 2) = 3$. Thus the nearest distance is $\sqrt{3}$ and the fastest is $\sqrt{9 + 2\sqrt{2}}$.