Partial Derivatives and Tangent Plane

1. Find the partial derivatives of the following functions, and find the tangent plane to the graph at the given point and a normal vector to the tangent plane.

   (a) $z = f(x, y) = 2x^2 - xy + y^2$ at the point $(1, -1, 4)$

   (b) $z = f(x, y) = (x^3 + \frac{y}{2})^{2/3}$ at the point $(0, -2, 1)$

   (c) $z = f(x, y) = e^{xy} \ln y$ at the point $(0, e, 1)$

   (d) $z = f(x, y) = \cos^2(3x - y^2)$ at the point $(0, 0, 1)$

2. Use the two variable chain rule to find df/dt for the following equations.

   (a) $f(x, y) = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$

   (b) $f(x, y) = -\sin xy$, $x = t$, $y = \ln t$

   (c) $f(x, y) = 2ye^x - \ln y$, $x = \ln(t^2 + 1)$, $y = \arctan t$

3. Suppose that we substitute polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ in a graph $z = f(x, y)$.

   (a) Give the chain rule for partial derivatives $\partial f/\partial r$ and $\partial f/\partial \theta$.

   (b) Show that $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta$ and $\frac{1}{r} \frac{\partial f}{\partial \theta} = -\frac{\partial f}{\partial x} \sin \theta + \frac{\partial f}{\partial y} \cos \theta$.

   (c) Show that $(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 = (\frac{\partial f}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial f}{\partial \theta})^2$.

Solutions:

1. (a) Tangent plane: $z = 5(x-1) - 3(y+1)$; normal vector: $\begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$.

   (b) $z = -\frac{1}{3}(y+2)$, $\begin{pmatrix} 0 \\ -1/3 \\ -1 \end{pmatrix}$.

   (c) $z = ex + \frac{1}{e}(y-e)$, $\begin{pmatrix} e \\ 1/e \\ -1 \end{pmatrix}$.

   (d) $z = 0$, $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

2. (a) 0

   (b) $-\cos(t\ln t)(\ln t + 1)$

   (c) $4t \arctan t + 2 - \frac{1}{(1+t^2)\arctan t}$

3. (a) $f_r = f_x x_r + f_y y_r = f_r = f_x \cos\theta + f_y \sin\theta$, $f_\theta = f_x x_\theta + f_y y_\theta = f_x(-r\sin\theta) + x_y(r\cos\theta)$.

   (b) Directly from part (a)

   (c) Use part (b)