Second Derivative Test & Lagrange Multipliers Method

1. Consider the quadratic polynomial $f(x, y) = ax^2 + bxy + cy^2$. If there is only one critical point, what condition should the coefficients a, b and c satisfy? In this case, use second derivative test to classify this critical point. What will happen if there are more than one critical point?

2. Let $f(x, y) = \sin(x)\sin(y)$.

   (a) Find the 2nd Taylor expansion (quadratic approximation) to f near the point $(x_0, y_0) = (\frac{\pi}{2}, \frac{\pi}{2})$.

   (b) Find all critical pints of f within the domain $-\pi < x < \pi$, $-\pi < y < \pi$.

(c) Classify each critical piont as a local minimum, a local maximum or s saddle point.

3. Suppose $f(x, y) = x^2 - y^2$.

   (a) Find the minimum and maximum values of f over the level set $g(x, y) = (x - 1)^2 + y^2 - 1$, which defines a bounded curve.

   (b) Are there any local minimum or maximum values of f over the level set $g(x, y) = (x - 1)^2 - y^2 = -3$? Can you explain the reason for your answer?

4. Consider a rectangle on the (x, y) plane whose bottom left corner is at the origin and whose bottom side and left side are on the x-axis and the y-axis, respectively. If its top right corner is on the ellipse $2x^2 + y^2 = 4$, use the Lagrange multiplier method to find the (x, y) coordinates of the top right corner for which the area of the rectangle is maximized.

Solutions:

1. If there is only one critical point, than $4ac \neq b^2$. In this case, the unique critical point is $(0,0)$. $(0,0)$ is a maximum point, if $4ac - b^2 > 0$ and $a < 0$; a minimum point, if $4ac - b^2 > 0$ and $a > 0$; and a saddle point if $4ac - b^2 < 0$. If there are more than one critical point, then $4ac - b^2 = 0$, which is the inconclusive case.

2. (a) $T_2 f(\frac{\pi}{2} + \Delta x, \frac{\pi}{2} + \Delta y) = 1 + \frac{1}{2}(-\Delta x^2 - \Delta y^2)$.

   (b) Saddle: $(0,0)$. Maxima: $(\pi/2, \pi/2)$ and $(-\pi/2, -\pi/2)$. Minima: $(-\pi/2, \pi/2)$ and $(\pi/2, -\pi/2)$.

3. (a) Maximun: $f(2,0) = 4$. Minima: $f(1/2, \pm\sqrt{3}/2) = -1/2$.

   (b) No maximum or minimum.

4. $f(x,y) = xy$. Maximum: $f(1, \sqrt{2}) = \sqrt{2}$.