Function of Two Variables

1. Let l be the line given by $\vec{x}(t) = (1+2t)\vec{i} + (t)\vec{j} + (3t+2)\vec{k}$. Find the unit tangent, the normal vector, curvature and the tangent line to the line l at the point where $t = 1$. Are all these answers the same at any point on the line?

2. Show that the curvature of a circle is the inverse of the radius.

3. Consider the curve given by $\vec{x}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + (t)\vec{k}$.
   (a) Find the unit tangent and tangent line to the curve at the point $t = 2\pi$.

(b) Find the normal vector and curvature at the point $t = 2\pi$.

(c) Find the binormal vector and the torsion to the curve at $t = 2\pi$, then give the equation of the osculating plane at this plane.

(d) Is there a unique circle in this 3-dimensional space that is tangent to this curve $\vec{x}(t)$ at $t = 2\pi$ and has the same curvature vector with $\vec{x}(t)$? Can you give the equation of this circle?

4. Consider the curve $\vec{x}(t) = (t)\vec{i} + (t^2)\vec{j}$. Give the function of curvature $c(t)$ with respect to the point t. Use the result you got in problem 2 and 3 to explain the meaning of $c(t)$.

Solutions:

1. $\vec{T}(1) = \begin{pmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$, tangent line: $\begin{pmatrix} 3+2t \\ 1+t \\ 5+3t \end{pmatrix}$, $\vec{N}(1) = \vec{0}$ and $\kappa(1) = 0$.

2.

3. (a) $\vec{T}(2\pi) = \begin{pmatrix} 0 \\ 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$, tangent line: $\begin{pmatrix} 0 \\ 3+3t \\ t \end{pmatrix}$.

   (b) $\vec{N}(2\pi) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\kappa(2\pi) = 3/10$.

   (c) $\vec{B}(2\pi) = \begin{pmatrix} 0 \\ -1/\sqrt{10} \\ -3/\sqrt{10} \end{pmatrix}$, $\tau(2\pi) = 1/10$, osculating plane $y + 3z - 6\pi = 0$.

   (d) This circle has radius $10/3$ and has center at $(-1/3, 0, 2\pi)$. So equation of this circle is the intersection of the ball $(x + \frac{1}{3})^2 + y^2 + (z - 2\pi)^2 = 100/9$ and the osculating plane $y + 3z - 6\pi = 0$.

4. $\kappa(t) = 2/(1 + 4t^2)^{3/2}$