Review: Vector Spaces

1. (a) Can we compute the cross product in a 2-dimensional space?

   (b) Is the cross product a number or a vector?

   (c) Is $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$? Is $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$?

   (d) What does $\vec{a} \cdot \vec{b} = 0$ imply? What does $\vec{a} \times \vec{b} = \vec{0}$ imply?

   (e) What's the relationship between $\vec{a}$ and $\vec{a} \times \vec{b}$.

2. (a) Compute the following cross products:

   i. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

   ii. $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

   iii. $\vec{a} \times \vec{b}$, where $\vec{a} = \vec{i} + \vec{j}$ and $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$.

3. **Triple Product:**

   (a) Compute $\vec{i} \cdot (\vec{j} \times \vec{k})$.

   (b) $\vec{a} \cdot (\vec{b} \times \vec{b})$.

   (c) $\vec{a} \cdot (\vec{a} \times \vec{b})$

4. **An Application of Cross Product** The cross product is always used to find a vector which is perpendicular to two given vectors, since $\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{b}$. Here is an example:

   Consider a plane in a 3-dimensional space, whose equation is $x + 2y + 4z = 3$. Find a vector perpendicular to this plane by the following steps:

   (a) Find 3 points A, B and C on this plane.

   (b) Compute the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

   (c) Compute the vector $\overrightarrow{AB} \times \overrightarrow{AC}$. And check that this vector is what we want.

   (d) In part(a), can these three points A, B and C lie on a line? Why?

Solutions:

1. (a) No
   (b) Vector
   (c) Yes. No.
   (d) $\vec{a} \perp \vec{b}$. $\vec{a} // \vec{b}$.
   (e) $\vec{a} \perp \vec{a} \times \vec{b}$

2. (a) i. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

   ii. $\begin{pmatrix} 4 \\ 4 \\ -8 \end{pmatrix}$

   iii. $\vec{i} - \vec{j} + \vec{k} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

3. (a) 1
   (b) 0
   (c) 0

4. (a) $A : (3, 0, 0)$, $B : (1, 1, 0)$ and $C : (-1, 0, 1)$.
   (b) $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$.
   (c) normal vector: $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$
   (d) No. Because if A, B and C lie on a line, then $\overrightarrow{AB} \times \overrightarrow{AC} = 0$.