Math 272—Spring 1998
More Extra Credit Problems.

1. a. Find the equation to the tangent lines to the circle \( x^2 + y^2 = 25 \) at those points where \( x = 4 \).

b. Compute the limit: \( \lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \)

2. Let \( f(x) = \frac{1}{x} \arcsin(2x - 4) \).
   a. Show that \( y = f(x) \) satisfies the equation \( \sin(xy) = 2x - 4 \).
   b. Compute \( f'(2) \) by direct calculation (i.e. by applying quotient, product, chain rules etc.)
   c. Compute \( f'(2) \) by implicit differentiation and compare your answer with (b).

3. Let \( S(x) \) be the number of sunlight hours on a cloudless June 21, as a function of the latitude \( x \) (measured in degrees).
   a. What is \( S(0) \)? (Hint: the equator is at zero degrees latitude.)
   b. Let \( x_0 \) be the latitude of the arctic circle (so \( x_0 = 66^{30'} \)). In the northern hemisphere \( S(x) \) is given by
      \[
      S(x) = \begin{cases} 
      24 & \text{for } x_0 \leq x \leq 90 \\
      a + b \arcsin \left( \frac{\tan x}{\tan x_0} \right) & \text{for } 0 \leq x \leq x_0
      \end{cases}
      \]
   c. Compute \( S(x) \) for
      Tucson, Arizona (\( x = 32^{13'} \)) and
      Walla Walla, Washington (\( x = 46^{4'} \)).

4. Water is being poured into a vase at a rate of 1 gallon per minute. Here is the vase:

   ![](image)

   Let \( V(t) \) be the volume of the water in the vase at time \( t \) and let \( h(t) \) be the height of the water level at time \( t \).
   a. What are the signs of the derivatives \( V'(t) \) and \( h'(t) \) and of the second derivatives \( V''(t) \) and \( h''(t) \) when the level \( h(t) \) is between \( h_1 \) and \( h_2 \)?
   b. Same question when the level \( h(t) \) is between 0 and \( h_1 \)?
   c. What is the level \( h(t) \) when \( h'(t) \) is maximal? At which level \( h(t) \) is \( h'(t) \) minimal?

5. Sketch the graphs of
   \[
   y = \sin(9x) + \sin(11x).
   \]
   and
   \[
   y = \sin(19x) + \sin(21x)
   \]
   (You could try using a graphing calculator first.) Explain your drawing using the trigonometric identity for \( \sin A + \sin B \).

6. Sketch the graphs of
   \[
   y = e^{-x} \sin(10x),
   \]
   \[
   y = e^{-x} \sin(x)
   \]
   and
   \[
   y = e^{-x/2} \sin(10x)
   \]
say, on the interval \(-1 \leq x \leq 8\). You could again try using a graphing calculator first.

   Now, consider the function \( f(x) = e^{-ax} \sin(bx) \) where \( a \) and \( b \) are constants.
   a. What is the distance between consecutive zeroes of \( f(x) \)?
   b. Find all local maxima of \( f(x) \). What is the distance between consecutive local maxima of \( f(x) \)?

7. How Big is \( n! \)?
   a. Compute \( J = \int_1^n \ln x \, dx \)
   b. What relation is there between \( n! \) and \( \sum_{k=1}^n \ln k \)?
   c. What are the left-hand and right-hand sums for \( \int_1^n \ln x \, dx \) if you divide the interval \( 1 \leq x \leq n \) into \( n \) pieces of equal length?
   d. By comparing the integral \( J \) with the left and right-hand sums (which is bigger, which is smaller?) show that
      \[
      n \ln n - n + 1 < \ln n! < (n+1) \ln n - n + 1
      \]
   e. How many decimals does the number 1000! have?