Math 832 – Spring 2013

Homework 7

Due: Thursday, May 16th at noon in my mailbox on the 2nd floor of Van Vleck.

1. Use the definition of the Itô integral to prove that

\[ \int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds. \]

To do this, you will need to provide an appropriate approximating sequence (in a correct space). Also, and as a comparison, derive the above expression using Itô’s formula.

2. Suppose that \( f(t, x) \in C^{1,2}(\mathbb{R}_{\geq 0} \times \mathbb{R}) \) (functions with 1 continuous derivative in \( t \) for all \( t \geq 0 \), and 2 continuous derivatives in \( x \)). Show, using arguments similar to those presented in class for Itô’s formula in the non-time dependent case, that

\[ f(t, B(t)) = f(0, 0) + \int_0^t \frac{\partial f}{\partial x}(s, B(s))dB(s) + \int_0^t \frac{\partial f}{\partial t}(s, B(s))ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B(s))ds. \]

As in class, just show the result for functions with compact support in both the time and space variable. The more general result follows from a localization argument that is currently outside the material presented in this course.

3. (a) Suppose that \( f \in C^{1,2}(\mathbb{R}^+, \mathbb{R}) \) and that \( f \) satisfies the PDE

\[ \frac{\partial f}{\partial t} = -\frac{1}{2} \frac{\partial^2 f}{\partial x^2}. \]

Let \( X(t) = f(t, B(t)) \) and suppose that

\[ \mathbb{E} \left[ \int_0^T \left\{ \frac{\partial f}{\partial x}(t, B(t)) \right\}^2 dt \right] < \infty. \]

Show that \( X(t) \) is a martingale on \( 0 \leq t \leq T \).

(b) Show that \( M_t = \exp\{\alpha B(t) - \alpha^2 t/2\} \) is a martingale using the above condition.

4. We say that \( X_t \) satisfies a stochastic differential equation

\[ dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \]

where \( b \) and \( \sigma \) are functions, if \( X_t \) satisfies the integral equation

\[ X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \]

which we have been studying. Show that

\[ N_t = N_0 \exp ((r - \alpha^2/2)t + \alpha B_t) \]

is a solution to the stochastic differential equation

\[ dN_t = rN_t dt + \alpha N_t dB_t. \]