NAME:

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(1) (10 points) For each of the following, give an example if one exists. If there is none, state that there is no example. You do not need to give any justification.

(a) A continous map from $C$ (the cantor set) onto $\mathbb{R}$.
   Solution: No example. $C$ is compact, $\mathbb{R}$ is not.

(b) A continous function and a compact set $A$ so that $f^{-1}(A)$ is not compact.
   Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ for every $x \in \mathbb{R}$. Then $A = \{0\}$ is an example.

(c) A nested sequence $A_0 \supseteq A_1 \supseteq A_2 \ldots$ of non-empty closed and bounded subsets of $\mathbb{R}^2$ with empty intersection.
   Solution: No example. Closed and bounded subsets of $\mathbb{R}^2$ are compact.

(d) A closed and bounded set $A$ in a complete metric space $M$ so that $A$ is not compact.
   Solution: $\mathbb{R}_{\text{disc}}$ (\mathbb{R} with the discrete metric) as a subset of itself.

(e) A finite collection of compact sets $A_i$ so that $A = \bigcup_i A_i$ is not compact.
   Solution: No example. A finite union of compact sets is compact. (Think about why - you can prove it fairly easily using either definition of compactness.)
(2) (10 points)

(a) Let $A$ be compact, $x \in A$. Let $(x_n)$ be a sequence in $A$ such that every convergent subsequence of $(x_n)$ converges to $x$. Show that $(x_n)$ converges to $x$.

(b) Show that every compact set is separable ($A$ is separable if there is a countable set $X \subseteq A$ so $A \subseteq \text{cl}(X)$).

Solution: These are homework problems 94 and Prelim 13 respectively. See the homework solutions for solutions.
(3) (10 points)

(a) Give the definition of uniform continuity.

(b) Let $f : M \to N$ be a function. Suppose $M = A \cup B$ and $B_1(x)$ is contained in $A$ or contained in $B$ for every point $x \in M$. Suppose $f|_A$ is uniformly continuous ($f|_A : A \to N$ is given by $f|_A(a) = f(a)$) and $f|_B$ is uniformly continuous. Show that $f$ is uniformly continuous.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(x) = 0$ for all $x \geq 100$ and $f(x) = 0$ for all $x \leq -100$. Show that $f$ is uniformly continuous. (Hint: Use part (b).)

Solutions:

(a) $f : M \to N$ is uniformly continuous if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in M \forall y \in M (d_M(x, y) < \delta \to d_N(f(x), f(y)) < \epsilon)$$

(b) Suppose $f$ is uniformly continuous on $A$ and $B$. Fix $\epsilon > 0$. There are $\delta_A$ and $\delta_B$ so that for any $x, y \in A$ where $d_M(x, y) < \delta_A$ or $x, y \in B$ where $d_M(x, y) < \delta_B$, the distance $d_N(f(x), f(y)) < \epsilon$. Let $\delta = \min\{\delta_A, \delta_B, 1\}$. Then if $d_M(x, y) < \delta \leq 1$ implies that either $x, y \in A$ or $x, y \in B$. If $x, y \in A$, then $d_M(x, y) < \delta \leq \delta_A$, so $d_N(f(x), f(y)) < \epsilon$. If $x, y \in B$, then $d_M(x, y) < \delta \leq \delta_B$, so $d_N(f(x), f(y)) < \epsilon$. So, we see that $d_M(x, y) < \delta$ implies $d_N(f(x), f(y)) < \epsilon$.

(c) Consider the two sets $A = [-102, 102]$ and $B = (-\infty, -100] \cup [100, \infty)$. Then $f$ is uniformly continuous on $A$, since $A$ is compact, and $f$ is uniformly continuous on $B$, since $f$ is constant on $B$. By part (b), we see that $f$ is uniformly continuous on all of $\mathbb{R}$. 

(4) (15 points) Show that any connected metric space \( M \) containing at least two points is uncountable. (Hint: Let \( a \) and \( b \) be two points in \( M \). Try to show that for any \( \alpha \in [0, 1] \), there is a point \( x \) so \( d(a, x) = \alpha d(a, b) \).)

Solution: Let \( \alpha \in (0, 1) \) be given. Suppose, towards a contradiction that there is no \( x \) so that \( d(a, x) = \alpha d(a, b) \). Then \( B_{\alpha d(a, b)}(a) = C_{\alpha d(a, b)} \). Thus, this is a clopen subset of \( M \). Also, \( a \) is in this set and \( b \) is not. So, it is a proper clopen subset of \( M \). This is a contradiction. So, for each \( \alpha \in (0, 1) \), there is an \( x \) so that \( d(a, x) = \alpha d(a, b) \). These must all be distinct \( x \)’s, so we have found \( \text{Card}([0, 1]) \) different \( x \) in \( M \). Thus \( M \) is uncountable.

Alternatively, and easier, apply the IVT to the function \( d(a, -) \). It hits 0 and it hits \( d(a, b) \), so it must hit every value in between.