Complex Numbers and the Complex Exponential for Differential Equations
1. Complex numbers

The equation \( x^2 + 1 = 0 \) has no solutions, because for any real number \( x \) the square \( x^2 \) is nonnegative, and so \( x^2 + 1 \) can never be less than 1. In spite of this it turns out to be very useful to assume that there is a number \( i \) for which one has

\[
(1) \quad i^2 = -1.
\]

Any complex number is then an expression of the form \( a + bi \), where \( a \) and \( b \) are old-fashioned real numbers. The number \( a \) is called the real part of \( a + bi \), and \( b \) is called its imaginary part.

Traditionally the letters \( z \) and \( w \) are used to stand for complex numbers.

Since any complex number is specified by two real numbers one can visualize them by plotting a point with coordinates \((a, b)\) in the plane for a complex number \( a + bi \). The plane in which one plot these complex numbers is called the Complex plane, or Argand plane.

You can add, multiply and divide complex numbers. Here’s how:

To add (subtract) \( z = a + bi \) and \( w = c + di \)

\[
z + w = (a + bi) + (c + di) = (a + c) + (b + d)i,
\]

\[
z - w = (a + bi) - (c + di) = (a - c) + (b - d)i.
\]

To multiply \( z \) and \( w \) proceed as follows:

\[
zw = (a + bi)(c + di)
\]

\[
= a(c + di) + bi(c + di)
\]

\[
= ac + adi + bci + bdi^2
\]

\[
= (ac - bd) + (ad + bc)i
\]

where we have use the defining property \( i^2 = -1 \) to get rid of \( i^2 \).

To divide two complex numbers one always uses the following trick.

\[
\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}
\]

\[
= \frac{(a + bi)(c - di)}{(c + di)(c - di)}
\]

Now

\[
(c + di)(c - di) = c^2 - (di)^2 = c^2 - d^2 i^2 = c^2 + d^2,
\]
Obviously you do not want to memorize this formula: instead you remember the trick, i.e. to divide \( c + di \) into \( a + bi \) you multiply numerator and denominator with \( c - di \).

For any complex number \( w = c + di \) the number \( c - di \) is called its complex conjugate. Notation:

\[
\bar{w} = c - di.
\]

A frequently used property of the complex conjugate is the following formula

\[
w\bar{w} = (c + di)(c - di) = c^2 - (di)^2 = c^2 + d^2.
\]

2. Argument and Absolute Value

For any given complex number \( z = a + bi \) one defines the absolute value or modulus to be

\[
|z| = \sqrt{a^2 + b^2},
\]

so \( |z| \) is the distance from the origin to the point \( z \) in the complex plane (see figure 1).

The angle \( \theta \) is called the argument of the complex number \( z \). Notation:

\[
\arg z = \theta.
\]

The argument is defined in an ambiguous way: it is only defined up to a multiple of \( 2\pi \). E.g. the argument of \(-1\) could be \( \pi \), or \(-\pi \), or \(3\pi \), or, etc. In general one says \( \arg(-1) = \pi + 2k\pi \), where \( k \) may be any integer.

From trigonometry one sees that for any complex number \( z = a + bi \) one has

\[
a = |z| \cos \theta, \quad b = |z| \sin \theta,
\]

so that

\[
|z| = |z| \cos \theta + i|z| \sin \theta = |z|(\cos \theta + i \sin \theta).
\]

and

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{b}{a}.
\]

Example: Find argument and absolute value of \( z = 2 + i \).

Solution: \( |z| = \sqrt{2^2 + 1} = \sqrt{5} \). \( z \) lies in the first quadrant so its argument \( \theta \) is an angle between 0 and \( \pi/2 \). From \( \tan \theta = \frac{1}{2} \) we then conclude \( \arg(2 + i) = \theta = \arctan \frac{1}{2} \).

3. Geometry of Arithmetic

Since we can picture complex numbers as points in the complex plane, we can also try to visualize the arithmetic operations “addition” and “multiplication.” To add \( z \) and \( w \) one forms the parallelogram with the origin, \( z \) and \( w \) as vertices. The fourth vertex then is \( z + w \). See figure 2.

To understand multiplication we first look at multiplication with \( i \). If \( z = a + bi \) then

\[
iz = i(a + bi) = ia + bi^2 = ai - b = -b + ai.
\]
Thus, to form $iz$ from the complex number $z$ one rotates $z$ counterclockwise by 90 degrees. See figure 3.

If $a$ is any real number, then multiplication of $w = c + di$ by $a$ gives

$$aw = ac + adi,$$

so $aw$ points in the same direction, but is $a$ times as far away from the origin. If $a < 0$ then $aw$ points in the opposite direction. See figure 4.

Next, to multiply $z = a + bi$ and $w = c + di$ we write the product as

$$zw = (a + bi)w = aw + biw.$$
$z = a + bi$

$\theta = \text{arg } z$

$|z| = \sqrt{a^2 + b^2}$

$zw = aw + biw$

$|zw| = |z| \cdot |w|$

$\arg(zw) = \arg z + \arg w$

**Figure 4.** Multiplication of a real and a complex number

**Figure 5.** Multiplication of two complex numbers

$a : b$, which is the same as for the shaded right triangle on the right, so we conclude that these two triangles are similar.

The triangle on the left is $|w|$ times as large as the triangle on the right. The two angles marked $\theta$ are equal.

Since $|zw|$ is the length of the hypothenuse of the shaded triangle on the left, it is $|w|$ times the hypothenuse of the triangle on the right, i.e. $|zw| = |w| \cdot |z|$.

The argument of $zw$ is the angle $\theta + \varphi$; since $\theta = \arg z$ and $\varphi = \arg w$ we get the following two formulas

\begin{align*}
(3) & \quad |zw| = |z| \cdot |w| \\
(4) & \quad \arg(zw) = \arg z + \arg w,
\end{align*}

i.e. when you multiply complex numbers, their lengths get multiplied and their arguments get added.

4. Applications in Trigonometry

**Unit length complex numbers.** For any $\theta$ the number $z = \cos \theta + i \sin \theta$ has length 1: it lies on the unit circle. Its argument is $\arg z = \theta$. Conversely, any complex number on the unit circle is of the form $\cos \phi$, where $\phi$ is its argument.
The Addition Formulas. For any two angles \( \theta \) and \( \phi \) one can multiply
\[ z = \cos \theta + i \sin \theta \] and \( w = \cos \phi + i \sin \phi \). The product \( zw \) is a complex number of absolute value \( |zw| = |z| \cdot |w| = 1 \cdot 1 \), and with argument \( \arg(zw) = \arg z + \arg w = \theta + \phi \). So \( zw \) lies on the unit circle and must be \( \cos(\theta + \phi) + i \sin(\theta + \phi) \). Thus we have
\[ (5) \quad (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) = \cos(\theta + \phi) + i \sin(\theta + \phi). \]

By multiplying out the left hand side we get
\[ \cos(\theta + \phi) + i \sin(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi + i(\sin \theta \cos \phi + \cos \theta \sin \phi) \]
from which we get the addition formulas for \( \sin \) and \( \cos \):
\[ \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \]
\[ \sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi \]

De Moivre’s formula. For any complex number \( z \) the argument of its square \( z^2 \) is \( \arg(z^2) = \arg(z \cdot z) = \arg z + \arg z = 2 \arg z \). The argument of its cube is \( \arg(z^3) = \arg(z \cdot z^2) = \arg(z) + \arg z^2 = \arg z + 2 \arg z = 3 \arg z \). Continuing like this one finds that
\[ (6) \quad \arg z^n = n \arg z \]
for any integer \( n \).

Applying this to \( z = \cos \theta + i \sin \theta \) you find that \( z^n \) is a number with absolute value \( |z^n| = |z|^n = 1^n = 1 \), and argument \( n \arg z = n\theta \). Hence \( z^n = \cos n\theta + i \sin n\theta \). So we have found
\[ (7) \quad (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta. \]
This is de Moivre’s formula.

For instance, for \( n = 2 \) his tells us that
\[ \cos 2 + i \sin 2\theta = (\cos \theta + i \sin \theta)^2 = \cos^2 \theta - \sin^2 \theta + 2i \cos \theta \sin \theta. \]
Comparing real and imaginary parts on left and right hand sides this gives you the double angle formulas \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta \) and \( \sin 2\theta = 2 \sin \theta \cos \theta \).

For \( n = 3 \) you get, using the binomial theorem, or Pascal’s triangle,
\[ (\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3i \cos^2 \theta \sin \theta + 3i^2 \cos \theta \sin^2 \theta + i^3 \sin^3 \theta \]
\[ = \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) \]
so that
\[ \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta \]
and
\[ \sin 3\theta = \cos^2 \theta \sin \theta - \sin^3 \theta. \]
In this way it is fairly easy to write down similar formulas for \( \sin 4\theta \), \( \sin 5\theta \), etc.

5. The Complex Exponential Function

We finally give a definition of \( e^{a+bi} \). First we consider the case \( a = 0 \), and we define
\[ (8) \quad e^{it} = \cos t + i \sin t. \]
for any real number \( t \).
Example. \( e^{\pi i} = \cos \pi + i \sin \pi = -1 \). This leads to Euler's famous formula
\[ e^{\pi i} + 1 = 0, \]
which combines the five most basic quantities in mathematics: \( e, \pi, i, 1, \) and \( 0 \).

**Reasons why the definition (8) seems a good definition.**

**Reason 1.** We haven’t defined \( e^{it} \) before and we can do anything we like.

**Reason 2.** Substitute \( it \) in the Taylor series for \( e^x \):
\[
e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots
\]
\[= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots
\]
\[= 1 - \frac{t^2}{2!} + i\frac{t^3}{3!} - \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots
\]
\[= \cos t + i \sin t.\]

This is not a proof, because before we had only proved the convergence of the Taylor series for \( e^x \) if \( x \) was a real number, and here we have pretended that the series is also good if you substitute \( x = it \).

**Reason 3.** As a function of \( t \) the definition (8) gives us the correct derivative. Namely, using the chain rule (i.e. pretending it still applies for complex functions) we would get
\[
\frac{de^{it}}{dt} = ie^{it}.
\]

Indeed, this is correct. To see this proceed from our definition (8):
\[
\frac{de^{it}}{dt} = \frac{d\cos t + i\sin t}{dt} = \frac{d\cos t}{dt} + i\frac{d\sin t}{dt} = -\sin t + i\cos t = i(\cos t + i\sin t)
\]

**Reason 4.** The formula \( e^x \cdot e^y = e^{x+y} \) still holds. Rather, we have \( e^{it+is} = e^{it}e^{is} \). To check this replace the exponentials by their definition:
\[
e^{it}e^{is} = (\cos t + i\sin t)(\cos s + i\sin s) = \cos(t+s) + i\sin(t+s) = e^{i(t+s)}
\]

\( e^{a+bi} \) in general. To define the complex exponential in general, we set
\[
e^{a+bi} = e^a \cdot e^{ib} = e^a(\cos b + i\sin b).
\]

One verifies as above in “reason 3” that this gives us the right behaviour under differentiation. Thus, for any complex number \( r = a + bi \) the function
\[
y(t) = e^{rt} = e^a(\cos bt + i\sin bt)
\]

satisfies
\[
y'(t) = \frac{de^{rt}}{dt} = re^{rt}.
\]

This is all you need to know to read the section in our textbook on second order equations in the presence of “imaginary roots.”
6. Other handy things you can do with complex numbers

6.1. Partial fractions. Consider the partial fraction decomposition

\[ \frac{x^2 + 3x - 4}{(x - 2)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 4} \]

The coefficient \( A \) is easy to find: multiply with \( x - 2 \) and set \( x = 2 \) (or rather, take the limit \( x \to 2 \)) to get

\[ A = \frac{2^2 + 3 \cdot 2 - 4}{2^2 + 4} = \cdots. \]

Before we had no similar way of finding \( B \) and \( C \) quickly, but now we can apply the same trick: multiply with \( x^2 + 4 \), \( x^2 + 3x - 4 \), and substitute \( x = 2i \). This makes \( x^2 + 4 = 0 \), with result

\[ \frac{(2i)^2 + 3 \cdot 2i - 4}{(2i - 2)} = 2iB + C. \]

Simplify the complex number on the left:

\[ \frac{(2i)^2 + 3 \cdot 2i - 4}{(2i - 2)} = -4 + 6i - 4 = -8 + 6i \]

\[ -2 + 2i = -2 + 2i \]

\[ \frac{(-8 + 6i)(-2 - 2i)}{(-2)^2 + 2^2} = \frac{28 + 4i}{8} = \frac{7}{2} + \frac{i}{2} \]

So we get \( 2iB + C = \frac{7}{2} + \frac{i}{2} \); since \( B \) and \( C \) are real numbers this implies

\[ B = \frac{7}{4}, \quad C = \frac{1}{2}. \]

6.2. Complex amplitudes. A harmonic oscillation is given by

\[ y(t) = A \cos(\omega t - \phi), \]

where \( A \) is the amplitude, \( \omega \) is the frequency, and \( \phi \) is the phase of the oscillation. If you add two harmonic oscillations with the same frequency \( \omega \), then you get another harmonic oscillation with frequency \( \omega \). You can prove this using the addition formulas for cosines, but there’s another way using complex exponentials. It goes like this.

Let \( y(t) = A \cos(\omega t - \phi) \) and \( z(t) = B \cos(\omega t - \theta) \) be the two harmonic oscillations we wish to add. They are the real parts of

\[ Y(t) = A \cos(\omega t - \phi) + i \sin(\omega t - \phi) = Ae^{i \omega t - i \phi} = Ae^{-i \phi} e^{i \omega t} \]

\[ Z(t) = B \cos(\omega t - \theta) + i \sin(\omega t - \theta) = Be^{i \omega t - i \theta} = Be^{-i \theta} e^{i \omega t} \]

Therefore \( y(t) + z(t) \) is the real part of \( Y(t) + Z(t) \) which is easy to compute:

\[ Y(t) + Z(t) = Ae^{-i \phi} e^{i \omega t} + Be^{-i \theta} e^{i \omega t} = (Ae^{-i \phi} + Be^{-i \theta}) e^{i \omega t}. \]

If you now do the complex addition

\[ Ae^{-i \phi} + Be^{-i \theta} = Ce^{-i \psi}, \]

i.e. you add the numbers on the left, and compute the absolute value \( C \) and argument \( -\psi \) of the sum, then we see that \( Y(t) + Z(t) = Ce^{i (\omega t - \psi)} \). Since we were looking for the real part of \( Y(t) + Z(t) \), we get

\[ y(t) + z(t) = A \cos(\omega t - \phi) + B \cos(\omega t - \theta) = C \cos(\omega t - \psi). \]
The complex numbers $Ae^{-i\phi}$, $Be^{-i\theta}$ and $Ce^{-i\psi}$ are called the complex amplitudes for the harmonic oscillations $y(t)$, $z(t)$ and $y(t) + z(t)$.

The recipe for adding harmonic oscillations can therefore be summarized as follows: **Add the complex amplitudes.**

7. Problems

(1) (a) Compute the following complex numbers by hand, and check the answers on your calculator.

(b) Draw all numbers in the complex (or “Argand”) plane (use graph paper or quad paper if necessary).

(c) Compute absolute value and argument of all numbers involved.

(i.e. draw $1 + 2i$, $2 - i$ and the product; the same for the other problems)

\[
(1 + 2i)(2 - i) \quad (1 + i)(1 + 2i)(1 + 3i)
\]

\[
i^2 \quad i^3
\]

\[
i^4 \quad \frac{1}{i}
\]

\[
\left(\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}\right)^2 \quad \frac{1}{1 + i}
\]

\[
\frac{5}{2 - i} \quad (\frac{1}{2} + \frac{i}{2}\sqrt{3})^3
\]

\[
e^{\pi i/3} \quad \sqrt{2}e^{3\pi i/4}
\]

\[
e^{\pi i} + 1 \quad e^{17\pi i/4}
\]

(2) In the following picture draw $2w$, $\frac{3}{4}w$, $iw$, $-2iw$, $(2 + i)w$ and $(2 - i)w$.

(Try to make a nice drawing, use a ruler.)

![Figure 6. Picture for problem 2.](image)

(3) Let $\theta$ and $\phi$ be two angles.

(a) What are the arguments of $z = 1 + i\tan \theta$ and of $w = 1 + i\tan \phi$?

(Draw both $z$ and $w$.)

(b) Compute $zw$. 

(c) What is the argument of \(zw\)?
(d) Compute \(\tan(\arg zw)\).

(4) Find formulas for \(\cos 4\theta\) and \(\sin 4\theta\) in terms of \(\cos \theta\) and \(\sin \theta\), by using de Moivre’s formula.

(5) (a) Verify that \(e^{-it} = \frac{1}{e^{it}}\).
(b) Show that
\[
\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}
\]

(6) Find \(A\) and \(\phi\) where
\[A \cos(t - \phi) = 12 \cos(t - \pi/6) + 8 \cos(t - \pi/4)\]

(7) The general solution of a second order linear differential equation contains expressions of the form \(Ae^{i\beta t} + Be^{-i\beta t}\). These can be rewritten as \(C_1 \cos \beta t + C_2 \sin \beta t\).

If \(Ae^{i\beta t} + Be^{-i\beta t} = 2 \cos \beta t + 3 \sin \beta t\), then what are \(A\) and \(B\)?