PRACTICE PROBLEMS FOR THE 1ST MIDTERM

Some problems are in the form “True or false: Papadam?” In such problems you are supposed to (1) find out if “Papadam” is a true statement or not, and (2) provide reasons (i.e. a proof) for your conclusion.

1. SETS

(1) Prove: if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$.
(2) By definition $A \setminus B = \{x \in A \mid x \notin B\}$. True or false: if $A \subseteq B$ and $C \subseteq D$, then $A \setminus C \subseteq B \setminus D$?
(3) By definition $A \setminus B = \{x \in A \mid x \notin B\}$. True or false: if $A \subseteq B$ and $C \subseteq D$, then $A \setminus D \subseteq B \setminus C$?

2. INEQUALITIES

(1) True or False: If $x$ and $y$ are real numbers not equal to $-1$, then $x < y$ implies $\frac{x}{x+1} < \frac{y}{y+1}$?
(2) Show that for any pair of numbers $x, y \in \mathbb{R}$ with $0 < x < y$ one has $\frac{x}{y} < \frac{x+1}{y+1}$.
(3) Show that for any pair of numbers $x, y \in \mathbb{R}$ with $0 < y < x$ one has $\frac{x}{y} > \frac{x+1}{y+1}$.
(4) True or False: If $x, y > 1$ are real numbers, then $x \leq y$ implies $\frac{x}{x-1} \leq \frac{y}{y-1}$?

3. THE REAL NUMBERS

(1) Let $E$ and $F$ be subsets of $\mathbb{R}$ for which it is known that for all $x \in E$ and all $y \in F$ one has $x \leq y$. Prove that $\sup E \leq \inf F$.
(2) True or false: if $E$ and $F$ are subsets of $\mathbb{R}$ for which for $x \in E$ and $y \in F$ implies $x < y$, then one has $\sup E < \inf F$. (Explain your answer of course.)
(3) Find $\sup E$ and $\inf E$ for $E = \{ \frac{2n}{n+2} : n \in \mathbb{N} \}$.
(4) Consider the set of real numbers $E = \{ \frac{1}{x} \mid 0 < x < 2 \}$. Find $\sup E$ and $\inf E$ if they exist.
(5) Show that for $E = \{ (1 + n^2)^{-1} \mid n = 1, 2, 3, \ldots \}$ one has $\inf E = 0$.
(6) Prove: if $0 \leq x < \frac{1}{n^2}$ for all $n \in \mathbb{R}$, then $x = 0$.
(7) By definition, a function $f : (a, b) \to \mathbb{R}$ is said to be bounded from above if there is a number $M \in \mathbb{R}$ such that $f(x) \leq M$ holds for all $x \in (a, b)$.
   (a) Is the function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ bounded from above?
   (b) Is $g : (0, 1) \to \mathbb{R}$ given by $g(x) = 1 - \frac{1}{x}$ bounded from above?
   (c) Is $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x - x^2$ bounded from above?

4. MATHEMATICAL INDUCTION

(1) Prove by induction that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.
(2) Show that $A_n = 1 + 4 + 7 + 10 + \cdots + (3n - 2)$ satisfies $A_n = \frac{3}{2}n^2 - \frac{1}{2}n$ for $n = 1, 2, 3, \ldots$
(3) Prove that $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{(n-1)^2} = 1 - \frac{1}{n}$.
(4) Prove Bernoulli’s inequality by induction, namely, show that if $x > -1$ then $(1 + x)^n > 1 + nx$ for $n = 1, 2, 3, \ldots$
(5) Prove by induction that
$$\frac{2^2 - 1}{2^2} \times \frac{3^2 - 1}{3^2} \times \cdots \times \frac{n^2 - 1}{n^2} = \frac{n + 1}{2n}.$$

5. INTEGRALS AND AREAS

The book has a large number of problems, which are listed with the homework assignments. In particular, §2.4 has problems 1–14, and (more important than doing all of 1–14), problem 17.

Concerning volumes, do problems 1–4 from §2.13.