1. Find the solution to
\[ \frac{dy}{dx} = e^{x-y}, \quad y(0) = 1. \]

*Hint:* This diffeq is separable since \( e^{x-y} = e^x/e^y \).

2. Find the solution of
\[ \tan x \frac{dy}{dx} + y = \sin(x), \quad y(\frac{\pi}{4}) = A. \]

**Solution:** This equation is linear first order. If you want to put it in standard form you must first divide by \( \tan x \), the coefficient of \( \frac{dy}{dx} \). You get
\[ \frac{dy}{dx} + \cos(x) \sin(x) y = \cos(x). \]

The integrating factor is \( m(x) = \sin(x) \), which you either “see” or you compute \( m(x) = e^{\int \cos(x) \sin(x) \, dx} \). After multiplying with \( m \) you get
\[ \sin(x) y = \int \sin(x) \cos(x) \, dx = \frac{1}{2}(\sin(x))^2 + C. \]

The condition \( y(\frac{\pi}{4}) = A \) implies
\[ \frac{1}{2} \sqrt{2} A = \frac{1}{4} + C \implies C = \frac{1}{2} \sqrt{2} - \frac{1}{4}, \]

and thus
\[ y = \frac{1}{2} \sin(x) + \frac{A}{2} \sqrt{2} - \frac{1}{4}. \]

Since \( \lim_{x \to \pi/2} \sin(x) = 1 \) one has
\[ L_{\pi/2} = \frac{1}{4} + A \frac{1}{2} \sqrt{2}. \]

On the other hand \( \lim_{x \to 0} \sin(x) = \sin(0) = 0 \), so \( L_0 \) does not exist unless \( \frac{1}{2} \sqrt{2} - \frac{1}{4} = 0 \); so, for \( A = \frac{1}{2} \sqrt{2} \), one has
\[ y = \frac{1}{2} \sin(x) \] and thus
\[ L_0 = \lim_{x \to 0} y = \lim_{x \to 0} \frac{1}{2} \sin(x) = 0 \] when \( A = \frac{1}{2} \sqrt{2} \),

and
\[ L_0 \text{ does not exist when } A \neq \frac{1}{2} \sqrt{2}. \]

3. (a) Find the solution of
\[ 2y''(x) + y'(x) + y(x) = e^x, \quad y(0) = 1, y'(0) = A. \]

(b) For which value of \( A \in \mathbb{R} \) does the solution you found in part (a) satisfy \( \lim_{x \to \infty} y(x) = 0? \)

**Solution:** (Done in class.) The method is as follows:

1. Solve the homogeneous equation.
2. Find a particular solution, preferably by “educated guessing,” but when nothing works apply “Variation of Constants.”
3. The general solution to the inhomogeneous equation is \( y = y_{\text{hom}} + y_{\text{particular}} \).
4. Determine undetermined constants so that your solution satisfies the initial conditions.
In this problem the numbers are not very nice. The homogeneous equation is \(2y'' + y' + y = 0\). Trying \(y = e^{rx}\) leads to 

\[2r^2 + r + 1 = 0\]

which has solutions

\[r_{\pm} = \frac{-1}{4} \pm \frac{i}{4} \sqrt{7} = \alpha \pm i\Omega.
\]

(When ugly numbers show up give them a name: \(\alpha = -1/4, \Omega = (\sqrt{7})/4\).)

Then

\[y_h = e^{\alpha x} \{ C_1 \cos (\Omega x) + iC_2 \sin (\Omega x) \}.
\]

To get a particular solution try \(y_p(x) = Ke^x\). You find \(K = \frac{1}{2}\), so the general solution to the inhomogeneous problem is

\[y(x) = \frac{1}{4} e^x + e^{\alpha x} \{ C_1 \cos (\Omega x) + iC_2 \sin (\Omega x) \},
\]

(with \(\alpha\) and \(\Omega\) as above.)

To match the initial conditions one computes

\[y(0) = \frac{1}{4} + C_1 = 1, \text{ whence } C_1 = \frac{3}{4}.
\]

Furthermore

\[y'(x) = (\alpha C_1 + \Omega C_2)e^{\alpha x}\cos (\Omega x) + (\alpha C_2 - \Omega C_1)e^{\alpha x}\sin (\Omega x)
\]

so that

\[y'(0) = \alpha C_1 + \Omega C_2.
\]

The condition \(y'(0) = 0\) then implies

\[C_2 = -\frac{\alpha}{\Omega} C_1 = -\frac{3\alpha}{4\Omega} = \frac{3}{4\sqrt{7}}.
\]

4. (a) Find the solution of

\[4y''(x) + Ay'(x) + y(x) = 0, \quad y(0) = 1, y'(0) = 0
\]

where \(A > 0\).

Solution: Trying \(e^{rx}\) leads to the characteristic equation \(4r^2 + Ar + 1 = 0\), with roots

\[r_{\pm} = \frac{-A \pm \sqrt{A^2 - 4}}{8}.
\]

If \(A > 4\) there are two real exponents, \(r_{\pm}\), if \(0 < A < 4\) then there are two complex exponents

\[r_{\pm} = \frac{-A}{8} \pm \frac{i}{8} \sqrt{16 - A^2}.
\]

Either way, the general solution is

\[y(x) = C_+ e^{r_+ x} + C_- e^{r_- x},
\]

so that

\[y(0) = C_+ + C_- = 1
\]

\[y'(0) = r_+ C_+ + r_- C_- = 0
\]

Thus \(C_- = 1 - C_+\) which implies \((r_+ - r_-)C_+ = -r_-\), i.e.

\[C_+ = \frac{-r_-}{r_+ - r_-}, \text{ and } C_- = 1 - C_+ = \frac{r_+}{r_+ - r_-}.
\]

so the solution is

\[y(x) = \frac{-r_- e^{r_+ x} + r_+ e^{-r_- x}}{r_+ - r_-}.
\]

This formula holds when \(r_+ \neq r_-\), i.e. when \(A \neq 4\) (for \(A < 4\) you are led to complex exponentials).

When \(A = 4\) the solution is \(y = (C_1 + C_2 x)e^{rx}\), where \(r = -A/8 = -1/2\). The initial conditions imply that \(C_1 = 1,\) and \(C_2 = -r\), so that

\[y(x) = (1 - rx)e^{rx} = (1 + \frac{x}{2})e^{-x/2}.
\]

(b) For which value of \(A > 0\) does the solution you found in part (a) “oscillate as \(x \to \infty\)” (if \(y(x)\) has infinitely many zeros in the interval \(0 < x < \infty\) we say \(y(x)\) oscillates, if there is a number \(x_0\) such that \(y(x) \neq 0\) for all \(x \geq x_0\) then we say the function does not oscillate.)
Solution: If \( A > 4 \), then the exponents \( r_{\pm} \) are real, and \( r_{-} < r_{+} < 0 \). Thus the solution vanishes if
\[
y(x) = \frac{-r_{-} e^{rx} + r_{+} e^{-rx}}{r_{+} - r_{-}} = 0, \text{ i.e. if } e^{(r_{+} - r_{-})x} = \frac{r_{+}}{r_{-}}. \Rightarrow x = \frac{1}{r_{+} - r_{-}} \ln \frac{r_{+}}{r_{-}}.
\]
i.e. the solution vanishes only once. Therefore it doesn’t oscillate.
If \( A = 4 \) then the solution vanishes when \( (1 + \frac{1}{2}) e^{-rx/2} = 0 \), i.e. again only once. The solution doesn’t oscillate.
If \( 0 < A < 4 \) then abbreviate \( r_{\pm} = -\alpha \pm i\Omega \), with \( \alpha = A/8 \) and \( \Omega = \frac{1}{8} \sqrt{16 - A^2} \). One finds
\[
y(x) = e^{\alpha x} \frac{(\alpha + i\Omega) e^{-i\alpha x} - (\alpha - i\Omega) e^{i\alpha x}}{2i\Omega} \text{ (now simplify the complex exponentials.)}
\]
This solution clearly has infinitely many zeros, and it oscillates.

5. Which functions of the form \( y = e^{rx} \), with \( r \in \mathbb{C} \) a possibly complex constant, satisfy the differential equation
\[
y''' - 2y'' + 3y = 0.
\]
[Another typo: try doing the same problem for \( y''' - 3y'' + 2y = 0 \).]
Solution: Substitute \( y = e^{rx} \) in the diffeq, and you find \( (r^3 - 2r^2 + 3) e^{rx} = 0 \). This will happen if and only if \( r^3 - 2r^2 + 3 = 0 \). This is a quadratic equation in \( r^2 \), which has solutions \( r^2 = 1 \pm i/2 \). So there are four values of \( r \) for which \( e^{rx} \) satisfies the diffeq, and they are
\[
r = \pm \sqrt{1 \pm i/2}.
\]
(You could try to write \( \sqrt{1 + i/2} \) as \( a + bi \).)

6. The general solution of the third order equation
\[
y''' - 7y'' + 6y = 0
\]
turns out to be
\[
y_{\text{hom}}(x) = C_1 e^{x} + C_2 e^{2x} + C_3 e^{-3x}.
\]
where \( C_1, C_2, C_3 \in \mathbb{R} \) are arbitrary constants. (This means: (i) for any choice of \( C_1, C_2, C_3 \in \mathbb{R} \) the function \( y_{\text{hom}} \) is a solution, and, (ii) if \( y \) is a solution then there exist \( C_1, C_2, C_3 \in \mathbb{R} \) such that \( y(x) = y_{\text{hom}}(x) \).

(a) Find a particular solution \( y_p \) of
\[
y''' - 7y'' + 6y = e^{-x}.
\]
[The original problem had two typos, namely it said “... – 7y’...” instead of “... – 7y’’...”, and the label (†) was put on the homogeneous equation instead of the inhomogeneous equation.]
Solution: Use the method of “educated guessing.” Try \( y_p = Ke^{-x} \). This gives
\[
-Ke^{-x} - 7Ke^{-x} + 6Ke^{-x} = e^{-x}, \text{ i.e. } -2Ke^{-x} = e^{-x}, \text{ i.e. } K = -\frac{1}{2}.
\]
(b) Prove that for every other solution \( y \) of (†) there exist constants \( C_1, C_2, C_3 \in \mathbb{R} \) such that \( y(x) = y_{\text{hom}}(x) + y_p(x) \).
Solution: Suppose \( y \) and \( y_p \) both satisfy (†). Then define \( z = y - y_p \). One has
\[
z''' - 7z'' + 6z = y''' - y_p''' - 7y'' + 7y''_p + 6y - 6y_p
\]
\[
= (y''' - 7y'' + 6y) - (y_p''' - 7y_p'' + 6y_p)
\]
\[
= 0 - 0 = 0
\]
so that \( z \) satisfies the homogeneous equation. We are given that this implies that \( z = C_1 e^{x} + C_2 e^{2x} + C_3 e^{-3x} \) for certain values of \( C_1, C_2, C_3 \in \mathbb{R} \). Since \( z = y - y_p \), we see that \( y = y_p(x) + C_1 e^{x} + C_2 e^{2x} + C_3 e^{-3x} \).

7. Let \( (a, b) \to \mathbb{R} \) be a solution of the differential equation
\[
y'(x) + P(x)y(x) = 0, \quad (a < x < b),
\]
where \( P : (a, b) \to \mathbb{R} \) is a continuous function. Suppose that for some \( c \in (a, b) \) one has \( y(c) = 0 \). Prove that \( y(x) = 0 \) for all \( x \in (a, b) \).
Solution: This equation is both separable and linear first order. Treating it as a separable equation \( y' = -P(x)y \) would lead you to divide by \( y \). For this problem that’s not good, since we presumably want to use \( y(c) = 0 \) at some point, so that division by \( y \) is division by zero. So for this problem we REJECT the method of separation of variables.

We treat the diffeq as a first order linear equation and look for an integrating factor. (See the corresponding section in the book also.) Let \( m(x) = e^{\int P(x) dx} \). Then \( m(x) \neq 0 \) for all \( x \in (a, b) \), since an exponential never vanishes. Moreover \( m(x) \) satisfies \( m'(x) = P(x)m(x) \). Now consider the function \( m(x)y(x) \). It satisfies

\[
\frac{dm(x)y(x)}{dx} = m(x)y'(x) + m'(x)y(x) = m(x) \{ y'(x) + P(x)y(x) \} = 0.
\]

We conclude that \( m(x)y(x) \) must be constant.

But at \( x = c \) we have \( y(c) = 0 \), so that \( m(x)y(x) = m(c)y(c) = 0 \) for all \( x \in (a, b) \).

Since \( m(x) \neq 0 \) for all \( x \in (a, b) \), we may divide by \( m(x) \) and we find that since \( m(x)y(x) = 0 \), one has \( y(x) = 0 \) for all \( x \in (a, b) \).