LINEAR FIRST ORDER
DIFFERENTIAL EQUATIONS

1. Solution Method

Suppose $A, B$ and $C$ are continuous functions on an interval $a < x < b$, and suppose $A(x) \neq 0$ for all $x \in (a, b)$.

To solve a differential equation of the form

$$A(x)\frac{dy}{dx} + B(x)y(x) = C(x),$$

you first divide both sides by $A(x)$, to get

$$\frac{dy}{dx} + P(x)y(x) = Q(x),$$

where $P(x) = B(x)/A(x)$ and $Q(x) = C(x)/A(x)$.

Next, you multiply the equation with a function $m : (a, b) \to \mathbb{R}$ which is to be determined later:

$$m(x)\frac{dy}{dx} + m(x)P(x)y(x) = m(x)Q(x),$$

We require that $m$ satisfies

$$\frac{dm}{dx} = m(x)P(x).$$

Any function $m(x)$ which satisfies this equation is called an integrating factor.

Equation (4) is a separable differential equation for $m$, so you can always solve it. One solution is given by

$$m(x) = e^{\int P(x)dx}$$

Let $m$ be an integrating factor. Then (3) implies

$$m(x)\frac{dy}{dx} + \frac{dm}{dx}y(x) = m(x)Q(x).$$

The left hand side here is precisely what you get if you differentiate $m(x)y(x)$ using the product rule, so we get

$$\frac{dm(x)y(x)}{dx} = m(x)Q(x).$$

Integrate to get

$$m(x)y(x) = \int m(x)Q(x) \, dx$$

and thus

$$y(x) = \frac{1}{m(x)} \int m(x)Q(x) \, dx.$$
2. The Homogeneous and the Inhomogeneous Equations

Consider the linear differential equation (1)

\[ A(x) \frac{dy}{dx} + B(x)y(x) = C(x) \]

again. This equation is called the inhomogeneous equation. The corresponding homogeneous equation is the equation you get by replacing the right hand side with 0. In other words, it’s

(10) \[ A(x) \frac{dy}{dx} + B(x)y(x) = 0. \]

There are two basic important features of linear (differential) equations which are summarized in the following two theorems.

**Theorem 1 (Superposition Principle).** Let \( y_1, y_2 : (a, b) \to \mathbb{R} \) be two solutions of the homogeneous equation (10). Then for any two real numbers \( \alpha \) and \( \beta \) the function \( y_3(x) = \alpha y_1(x) + \beta y_2(x) \) is again a solution of the homogeneous differential equation (10).

**Theorem 2 (About Particular Solutions).**

(i) If \( y_h : (a, b) \to \mathbb{R} \) is a solution of the homogeneous equation (10), and if \( y_p : (a, b) \to \mathbb{R} \) is a solution of the inhomogeneous equation (1), then the sum \( y(x) = y_p(x) + y_h(x) \) is also a solution of the inhomogeneous equation (1).

(ii) If \( y_1, y_2 : (a, b) \to \mathbb{R} \) are two solutions of the inhomogeneous equation (1), then their difference \( y_h(x) = y_1(x) - y_2(x) \) is a solution of the homogeneous equation.

You should know the proofs of these theorems! They will be given in lecture.

{One reason why these theorems are important is that they apply to many, many more linear differential equations (e.g. Schrödinger’s equation in quantum mechanics is a linear differential equation for the wavefunction, and the superposition principle for that equation is what allows Quantum Mechanical particles to be in more than one state at a time.)

Another reason is that they might be on the next midterm.}

The second theorem gives an alternative strategy for solving the inhomogeneous equation. Namely, first you find any old solution \( y_p(x) \) of the inhomogeneous equation (it doesn’t matter how, sometimes there’s one obvious solution that stands out…). You call this solution a “particular solution.” Next you solve the homogeneous equation, i.e. you find all solutions of (10). Theorem 2 then says that if \( y_h(x) \) is your solution to the homogeneous equation, then the general solution to the inhomogeneous equation is given by

(11) \[ y_{\text{inhom}}(x) = y_h(x) + y_p(x). \]