SECOND ORDER
LINEAR DIFFERENTIAL EQUATIONS

1. Solving 2 × 2 systems of linear equations

   From algebra you know how to solve a linear system of equations

   \[
   \begin{cases}
   ax + by = p \\
   cx + dy = q
   \end{cases}
   \]

   in two unknowns \(x\) and \(y\). For instance, you could multiply the first equation with \(d\) and the second with \(c\), and subtract, with result \((ad - bc)x = pd - qc\). This gives you \(x\). A similar trick will give you \(y\). In the end the solution is given by

   \[
   x = \frac{dp - bq}{ad - bc}, \quad y = \frac{aq - cp}{ad - bc}.
   \]

   There is a special notation for the quantity \(ad - bc\) which occurs in the denominator. It is called the determinant of the system, and is written as

   \[
   \begin{vmatrix}
   a & b \\
   c & d
   \end{vmatrix} = ad - bc.
   \]

   With this notation we can reformulate the above as follows

   **Theorem 1** (“Cramer’s rule,” the 2 × 2 case). If \(\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0\) then the system of equations

   \[(1)\]

   has a solution for any given \(p, q \in \mathbb{R}\). This solution is given by

   \[
   x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.
   \]

2. The Wronskian and Abel’s theorem

   Consider the second order linear differential equation

   \[(4)\]

   \[
   y''(x) + a(x)y'(x) + b(x)y(x) = f(x),
   \]

   and its associated **homogeneous equation**

   \[(5)\]

   \[
   y''(x) + a(x)y'(x) + b(x)y(x) = 0.
   \]

   Here, and in the following all functions are assumed to be defined on some interval \(x_1 < x < x_2\).

   **Definition.** If \(y_1\) and \(y_2\) are solutions of the homogeneous equation, then their **Wronskian** is defined to be the function

   \[
   W(x) \overset{\text{def}}{=} W(y_1, y_2; x) \overset{\text{def}}{=} y_1(x)y_2'(x) - y_1'(x)y_2(x).
   \]
Using the determinant notation we have therefore defined the Wronskian to be
\[
W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix}.
\]

**Theorem 2** (Niels Henrik Abel, (1802–1829)). The Wronskian of two solutions of the linear homogeneous differential equation (5) satisfies
\[
\frac{dW}{dx} = -a(x)W(x).
\]

Hence \(W(x)\) is given by
\[
W(x) = W(x_0)e^{\int_{x_0}^x a(x')dx'}.
\]

3. The Method of Variation of Constants

To solve the inhomogeneous equation (4) one can use the method of *Variation of Constants* (or “variation of parameters”). In this method one assumes that the solution \(y\) is given by
\[
y(x) = c_1(x)y_1(x) + c_2(x)y_2(x),
\]
and that the functions \(c_1\) and \(c_2\) satisfy
\[
y'(x) = c_1(x)y'_1(x) + c_2(x)y'_2(x),
\]
Such functions \(c_1\) and \(c_2\) always exist, at least if the Wronskian of the two solutions \(y_1\) and \(y_2\) is nonzero (see problem 4). If this is so, then one has
\[
y''(x) = c'_1(x)y'_1(x) + c'_2(x)y'_2(x) + c_1(x)y''_1(x) + c_2(x)y''_2(x)
\]
and
\[
a(x)y'(x) = a(x)c_1(x)y'_1(x) + a(x)c_2(x)y'_2(x)
\]
\[
b(x)y(x) = b(x)c_1(x)y_1(x) + b(x)c_2(x)y_2(x)
\]

Keep in mind that \(y_1\) and \(y_2\) both satisfy the homogeneous equation, and add vertically. You find that
\[
f(x) = c'_1(x)y'_1(x) + c'_2(x)y'_2(x).
\]

This gives us one equation for \(c'_1(x)\) and \(c'_2(x)\). To get a second equation we differentiate (7), applying the product rule, and combine the result with (8). One gets
\[
0 = c'_1(x)y_1(x) + c'_2(x)y_2(x).
\]
Equations (9) and (10) together form a system of two equations for the unknowns \(c'_1(x)\) and \(c'_2(x)\), namely
\[
\left\{ \begin{array}{l} y_1(x)c'_1(x) + y_2(x)c'_2(x) = 0 \\ y'_1(x)c'_1(x) + y'_2(x)c'_2(x) = f(x) \end{array} \right.
\]
If the Wronskian \(W(x) = y'_1(x)y_2(x) - y_1(x)y'_2(x)\) is nonzero, then one can solve this system for \(c'_1(x)\) and \(c'_2(x)\). Integrating \(c'_1(x)\) and \(c'_2(x)\) then gives \(c_1(x)\) and \(c_2(x)\), and from there you get the solution \(y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)\).

When you work this out, you get
\[
c'_1(x) = \frac{-y_2(x)f(x)}{W(x)}, \quad c'_2(x) = \frac{y_1(x)f(x)}{W(x)},
\]
where $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of $y_1$ and $y_2$. Thus the solution of the inhomogeneous equation (4) is given by

$$y(x) = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} \, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} \, dx.$$  \hfill (13)

Both indefinite integrals contain a constant, so the general solution we have found has two undetermined constants in it.

Besides giving us a method for solving the inhomogeneous equation, this computation also lets us prove a uniqueness theorem for the homogeneous equation.

**Theorem 3.** Let $y_1, y_2 : (x_1, x_2) \to \mathbb{R}$ be two solutions of the homogeneous equation (5), for which the Wronskian $W(x)$ does not vanish. Then the general solution to the homogeneous equation (5) is

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x).$$  \hfill (14)

**Proof.** The homogeneous equation is just a special case of the inhomogeneous equation where $f(x)$ happens to vanish. So we can apply the method of Variation of Constants to get the general solution to the homogeneous equation by setting $f = 0$ in (13). The two integrals that appear in (13) now are:

$$\int y_2(x)f(x) \frac{dx}{W(x)} = \int 0 \, dx = C_1, \quad \int y_1(x)f(x) \frac{dx}{W(x)} = \int 0 \, dx = C_2.$$

Hence (13) says that the general solution is indeed given by (14). \hfill \Box

4. **Linearity and the Superposition Principle**

We abbreviate the lefthand side of the differential equation (4) by

$$\mathcal{L}[y] = y''(x) + a(x)y'(x) + b(x)y(x).$$

Thus the equations (4) and (5) can be written concisely as follows:

- inhomogeneous: $\mathcal{L}[y] = f$
- homogeneous: $\mathcal{L}[y] = 0$.

The expression $\mathcal{L}[y]$ is “linear in $y$,” which, by definition, means that for any two functions $y_1$ and $y_2$, and any two numbers $c_1$ and $c_2$ one has

$$\mathcal{L}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{L}[y_1] + c_2 \mathcal{L}[y_2].$$  \hfill (15)

Just as for 1st order equations one has a Superposition Principle.

**Theorem 4** (Superposition Principle).  

- (i) If $y_1$ and $y_2$ are solutions of the homogeneous equation, then so is any linear combination $y = c_1 y_1 + c_2 y_2$ ($c_1, c_2 \in \mathbb{R}$).

- (ii) If $y_1$ and $y_2$ are solutions of the inhomogeneous equations $\mathcal{L}[y_1] = f_1$ and $\mathcal{L}[y_2] = f_2$ respectively, then the linear combination $y = c_1 y_1 + c_2 y_2$ ($c_1, c_2 \in \mathbb{R}$ constants) satisfies $\mathcal{L}[y] = c_1 f_1 + c_2 f_2$.

- (iii) If $y_1$ and $y_2$ are solutions to the same inhomogeneous equation, i.e. if $\mathcal{L}[y_1] = f$ and $\mathcal{L}[y_2] = f$, then their difference $y_h = y_1 - y_2$ satisfies the homogeneous equation: $\mathcal{L}[y_h] = 0$.  

5. **Constant coefficient equations**

There is no formula that gives you the general solution to the homogeneous equation for an arbitrary second order linear equation. But if the coefficients \( a(x) \) and \( b(x) \) are constant, such a formula does exist.

Consider the differential equation

\[
y'' + py' + qy = 0,
\]

where \( p, q \in \mathbb{R} \) are constants. To solve this equation one looks for exponential functions which satisfy the equation. So set \( y = e^{rx} \) for some constant \( r \), and see if (16) holds:

\[
y'' + py' + qy = r^2 e^{rx} + pre^{rx} + qe^{rx} = (r^2 + pr + q)e^{rx}.
\]

Since \( e^{rx} \neq 0 \) no matter what \( r \) and \( x \) are (even if they are complex numbers!) we see that \( y = e^{rx} \) is a solution of the homogeneous equation if and only if \( r \) satisfies the quadratic equation

\[
r^2 + pr + q = 0.
\]

There are now three cases:

- \([p^2 - 4q > 0]\) In this case the characteristic equation has two real roots, \( r_1 \) and \( r_2 \), and we get two solutions \( y_1(x) = e^{r_1x} \) and \( y_2(x) = e^{r_2x} \) of the homogeneous equation. It follows from the superposition principle that

  \[
y(x) = c_1 e^{r_1x} + c_2 e^{r_2x}
\]

  is a solution of the homogeneous equation for any \( c_1, c_2 \in \mathbb{R} \).

- \([p^2 - 4q < 0]\) In this case the characteristic equation has two complex roots, which we write as

  \[
  r_{\pm} = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm i \sqrt{4q - p^2}}{2} = \alpha \pm i \Omega,
  \]

  where \( \alpha = -p/2 \) and \( \Omega = \frac{1}{2} \sqrt{4q - p^2} \) are real numbers, and \( \Omega > 0 \). The solutions in exponential form are now

  \[
y_{\pm}(x) = e^{r_{\pm}x} = e^{\alpha(x \pm i \Omega)} = e^{\alpha x} e^{\pm i \Omega x} = e^{\alpha x} (\cos \Omega x \pm i \sin \Omega x)
\]

  These solutions are complex valued. To get real valued solutions one forms these linear combinations:

  \[
y_1(x) = \frac{1}{2} (y_{+}(x) + y_{-}(x)) = e^{\alpha x} \cos \Omega x
\]

  \[
y_2(x) = \frac{1}{2i} (y_{+}(x) - y_{-}(x)) = e^{\alpha x} \sin \Omega x
\]

  Thus we find the following solutions for the homogeneous equation in this case:

  \[
y(x) = Ay_1(x) + By_2(x) = e^{\alpha x} (A \cos \Omega x + B \sin \Omega x).
\]

- \([p^2 - 4q = 0]\) In this last case the characteristic equation has one double root, namely \( r = -p/2 \). There is therefore only one exponential function \( y(x) = e^{rx} \) which satisfies the equation. It turns out that in this case there is another solution which is not exponential, namely, \( xe^{rx} \). So in this case we have the following solution to our constant coefficient equation (16),

  \[
y(x) = c_1 e^{rx} + c_2 xe^{rx} = (c_1 + c_2 x)e^{rx}.
\]
6. Questions and Problems

1. Derive the solutions in (2) (i.e. check those formulas.)

2. Solve
   \[
   \begin{aligned}
   x + iy &= 2 + 6i \\
   ix + y &= -4
   \end{aligned}
   \quad \text{and} \quad
   \begin{aligned}
   (2 + i)x - 2y &= 2 + 6i \\
   -2x + (2 - i)y &= -4
   \end{aligned}
   \]
   “by hand,” and again using Cramer’s rule.

3. Prove Abel’s theorem: verify that the Wronskian really does satisfy \(W'(x) = -a(x)W(x)\).

4. Suppose \(y_1\) and \(y_2\) are two solutions of the homogeneous equation (5) whose Wronskian does not vanish. Show: If \(y\) is an arbitrary differentiable function, then there always exist functions \(c_1(x)\) and \(c_2(x)\) such that (7) and (8) hold. (Can you write down a formula for \(c_1\) and \(c_2\) in terms of \(y, y', y_1, y_2, y'_1\) and \(y'_2\)?)

5. Which are the known functions, and which are the unknown functions in (11)?

6. Prove that (10) does indeed follow from the assumptions (7) and (8).

7. (i) State the definition of the statement \(\mathcal{L}[y]\) is linear in \(y\).”
   (ii) Show that the expression \(\mathcal{L}[y] = y''(x) + a(x)y'(x) + b(x)y(x)\) is indeed linear in \(y\).

8. Consider the operator \(\mathcal{M}\) defined by \(\mathcal{M}[y] = \frac{dy}{dx} + y^2\).
   (i) Compute \(\mathcal{M}[y]\) when \(y\) is the function \(y(x) = \sin x\). Do the same for \(y = 2\sin x\).
   (ii) Is \(\mathcal{M}[y]\) linear in \(y\)?

9. Prove Theorem 4!!

10. Find the general solutions to the following diffeqs:
    \[
    \begin{aligned}
    2y''(x) + 3y'(x) + y(x) &= 0 \\
    y''(x) - 16y(x) &= 0 \\
    y''(x) + Ay'(x) + y(x) &= 0 \\
    y''(x) - y'(x) + Ay(x) &= 0
    \end{aligned}
    \]
    where \(A > 0\) is some constant.

11. Explain how you can use the Superposition Principle (Theorem 4) to find a particular and from there the general solution to the differential equations
    \[
    \begin{aligned}
    y''(x) + \omega^2y(x) &= 1 + x + x^2 \\
    y''(x) + 2y'(x) - y(x) &= x^2 + \sin(Ax) \\
    6y''(x) + 5y'(x) + y(x) &= e^{Ax}
    \end{aligned}
    \]
    Here \(A\) and \(\omega\) are positive constants.

12. Use Variation of Constants to find the general solution of the following equations:
    \[
    \begin{aligned}
    y''(x) - y(x) &= x \\
    y''(x) + 4y(x) &= \sin Ax \\
    y''(x) - y'(x) &= e^{iAx} \quad \text{A > 0 is some constant.} \\
    y''(x) - y'(x) &= e^{i\omega x} \quad \text{\(\omega > 0\) is a constant.}
    \end{aligned}
    \]