GEODESIC CURVATURE PROBLEMS

The following problems require you to review the definition of geodesic curvature of a curve $\gamma$ on a surface. The most straightforward formula for $\kappa_g$ in this context is

$$\kappa_g = \vec{\kappa} \cdot (\vec{n} \times \vec{T}) = \frac{\gamma''(t) \cdot (\vec{n} \times \vec{T})}{\|\gamma'(t)\|^3}$$

$\vec{n}$ being the surface normal.

(1) Let $\gamma$ be the “small circle” on the unit sphere

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

obtained by intersecting $S$ with the plane $\{z = a\}$, where $a \in (-1, 1)$ is a constant.

(a) Find a surface patch $\sigma$ for $S$, and in this surface patch find a parametrization for $\gamma$ (suggestion: spherical coordinates, i.e. latitude & longitude probably work best.)

(b) Compute the geodesic curvature $\kappa_g$ at any point of the curve $\gamma$.

(c) For which values of $a$ does the curve $\gamma$ have zero geodesic curvature?

(2) Let $\mathcal{C}$ be the cylinder

$$\mathcal{C} = \{(x, y, z) \mid x^2 + y^2 = 1\}$$

and let

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ at \end{pmatrix}$$

be a helix on $\mathcal{C}$ ($a > 0$ is some constant.)

Compute the geodesic curvature of $\gamma$.

(3) On the same cylinder $\mathcal{C}$ as in the previous problem we consider the curve

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ h(t) \end{pmatrix},$$

where $h : \mathbb{R} \to \mathbb{R}$ is some smooth function.

(a) Find the geodesic curvature of $\gamma$.

(b) Show that the geodesic curvature of $\gamma$ vanishes if and only if $h(t) = at + b$ for certain constants $a$ and $b$.

(4) Suppose a curve $\gamma$ on a surface $S \subset \mathbb{R}^3$ has zero geodesic curvature, i.e. $\kappa_g = 0$. Must $\gamma$ be a straight line?

(5) Suppose a curve $\gamma$ on a surface $S \subset \mathbb{R}^3$ has zero geodesic curvature, and zero normal curvature, (so $\kappa_g = \kappa_n = 0$ on the curve). Must $\gamma$ be a straight line?

(6) Suppose a surface $S \subset \mathbb{R}^3$ contains a straight line $\gamma \subset S$. Show that $\gamma$ is a geodesic (i.e. a curve whose geodesic curvature vanishes.)