MATH 561 – Practice problems for the 2nd midterm

(1) The first fundamental form of a surface patch \( \sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) is given by
\[
(ds)^2 = (1 + v^2)(du)^2 - 2uv
du
dv + (1 + u^2)(dv)^2.
\]
(a) Compute the angle of intersection between the curves \( u = \text{constant} \) and \( v = \text{constant} \) at an arbitrary point \((u, v)\).
(b) What is the area of the image under \( \sigma \) of the rectangle \( \mathcal{U} = \{(u, v) \mid a < u < b, c < v < d\} \).
\(a, b, c, d\) are finite constants, you may leave an integral in the answer.

(2) Consider the helicoid, with coordinate patch \( \sigma(u, v) = \begin{pmatrix} u \cos(Av) \\ u \sin(Av) \\ Av \end{pmatrix} \)
where \( A > 0 \) is a constant.
(a) Compute the first and second fundamental forms of \( \sigma \).
(b) Compute the Gauss Curvature and the Mean Curvature of \( \sigma \).

(3) \textbf{A proof:} If \((ds)^2 = E(u, v)(du)^2 + 2F(u, v)du
dv + G(u, v)(dv)^2\) is the first fundamental form of some surface patch, then show that
\[ EG - F^2 > 0 \]
at all points of the surface patch.

(4) Let \( \mathcal{U} = \{(u, v) \mid u > 0, -\frac{\pi}{2} < v < \frac{\pi}{2}\} \) and consider the surface patch \( \sigma : \mathcal{U} \rightarrow \mathbb{R}^3 \) given by
\[ \sigma(u, v) = \begin{pmatrix} u \cos(Av) \\ u \sin(Av) \\ u \end{pmatrix} = u \begin{pmatrix} \cos(Av) \\ \sin(Av) \\ 1 \end{pmatrix} \]
in which \( A > 0 \) is a positive constant.
Compute 1st and 2nd fundamental forms, as well as the principal, principal curvature directions of \( \sigma \).

(5) Let \( \mathcal{U} = \{(u, v) \mid u > 0, -\pi < v < \pi\} \) and consider the surface patch \( \sigma : \mathcal{U} \rightarrow \mathbb{R}^3 \) given by
\[ \sigma(u, \theta) = \begin{pmatrix} Au \cos(\theta) \\ Bu \sin(\theta) \\ u \end{pmatrix} \]
where \( A \) and \( B \) are positive constants.
Compute 1st and 2nd fundamental forms, as well as the principal, principal curvature directions of \( \sigma \).

(6) \textbf{You will need the Serret-Frenet formulas in this problem (hence you should know them).}
Let \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^3 \) be an arclength parametrization of a space curve whose curvature is never zero.
Consider the surface patch given by
\[ \sigma(s, u) = \gamma(s) - u
d_s
d\gamma(s) \]
\( = \gamma(s) - u\gamma'(s) \).
(a) Show that \( \sigma : \mathcal{U} \rightarrow \mathbb{R}^3 \) is a surface patch if we choose its domain to be \( \mathcal{U} = \{(u, s) \in \mathbb{R}^2 \mid u > 0\} \).
(b) Compute the 1st and 2nd fundamental forms, as well as the principal curvatures, Gaussian curvature and Mean Curvature of \( \sigma \).

Comments: Since the curve \( \gamma \) is arbitrary, you don’t know its curvature \( \kappa(s) \) or torsion \( \tau(s) \), and hence these quantities and their derivatives with respect to \( s \) may show up in your work.
Their is a potential for confusion in the meaning of the variable \( s \). In this problem we use \( s \) for arc length along the curve \( \gamma \). One should then not write \((ds)^2\) for the 1st fundamental form of the surface patch \( \sigma \), since the \( s \) in \((ds)^2\) stands for arclength along any other curve on \( \sigma \). Instead, write “I” for the 1st form.