REVIEW PROBLEMS
for the math 561 final

CURVES

(1) The curvature of a plane curve as a function of arclength is given by \( \kappa(s) = \frac{1}{2\sqrt{s}} \). Find an arclength parametrization of the curve. (You may assume that at \( s = 0 \) the curve passes through the origin, and is tangent to the \( x \)-axis.)

(2) Consider the Helix

\[ \gamma(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + at \mathbf{k} \]

(a) Find the tangent line to the helix at the point \( \gamma(t) \).
(b) Let \( \alpha(t) \) be the point where this tangent line intersects the \( xy \) plane. The vector function \( t \mapsto \alpha(t) \) defines a plane curve. For which \( t \) is \( \alpha \) a regular curve?
(c) Compute the curvature of \( \alpha \) at all its regular points.

(3) Consider the space curve \( \gamma(t) = 6t \mathbf{i} + 3t^2 \mathbf{j} + t^3 \mathbf{k} \), and compute the angle its tangent makes with the vector \( \mathbf{w} = \mathbf{i} + \mathbf{k} \).

(4) Let \( \gamma : I \to \mathbb{R}^3 \) be a regular parametrized curve.

(a) Show that \( \cos \theta = \frac{dz}{ds} \) where \( \theta \) is the angle between the tangent to the curve and the \( z \)-axis, and \( \frac{dz}{ds} \) is the derivative w.r.t. arclength of the \( z \) component of \( \gamma \).

(b) Compute

\[ A = \frac{d\gamma}{ds} \left\{ \frac{d^2\gamma}{ds^2} \times \frac{d^3\gamma}{ds^3} \right\}, \quad \text{and} \quad B = \frac{d^4\gamma}{ds^4} \left\{ \frac{d^2\gamma}{ds^2} \times \frac{d^3\gamma}{ds^3} \right\}, \]

i.e. express these quantities in terms of the curvature, torsion and their derivatives.

SURFACES

(1) Find the first and second fundamental forms of the surfaces parametrized by

\[ \sigma(u,v) = u \mathbf{i} + v \mathbf{j} + f(u,v) \mathbf{k} \quad (f \text{ is any smooth function.}) \]

(2) State the definition of normal and geodesic curvatures of a curve on a surface.

(3) A curve \( \gamma \) passes through a point \( P \) on a surface \( S \). At this point, the normal to (the tangent plane of) the surface is \( \mathbf{n} = \frac{1}{2} \mathbf{i} + \frac{1}{2} \mathbf{j} + \frac{1}{\sqrt{2}} \mathbf{k} \), and the curvature vector of the curve is \( \mathbf{\kappa} = \mathbf{i} - \mathbf{k} \). Compute the normal and geodesic curvatures of the curve at the point \( P \).

(4) The helix \( \gamma(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + at \mathbf{k} \) lies on a surface \( S \). All we know about the surface is that the helix \( \gamma \) is a geodesic on this surface. Compute the normal \( \mathbf{n} \) to the surface at any given point on the helix.

(5) State the definition of the principal curvatures, and of the Gauss and Mean curvatures of a surface. (What is the relation between them?)

(6) Compute the principal curvatures and principal curvature directions of the surface

\[ \sigma(u,v) = u \mathbf{i} + v \mathbf{j} + (\cos u - v \sin v) \mathbf{k} \]

at the point \( u = v = 0 \).

(7) A surface \( S \) contains a curve \( \gamma \) whose normal curvature is zero. Show that the Gauss curvature of the surface is nonpositive along \( \gamma \).