“Superficial” Problem Set

PROBLEM 1

Let \( \Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \) be the unit sphere. Consider the surface patch

\[
\sigma(u, \theta) = \begin{pmatrix}
\cos \theta \\
\frac{\cosh u}{\cosh u}
\sin \theta \\
\frac{\tanh u}{\cosh u}
\end{pmatrix}, \quad (u, \theta) \in S = \{(u, \theta) \in \mathbb{R}^2 \mid u \in \mathbb{R}, |\theta| < \pi\}
\]

for the unit sphere.

(1) Compute the metric \((ds)^2 = E(u, \theta)(du)^2 + 2F(u, \theta)dud\theta + G(u, \theta)(d\theta)^2\).

(2) Show that the mapping from the strip \( S \) to the unit sphere \( \Sigma \) given by \( \sigma \) is conformal.

(3) Find a curve \( \gamma : (0, \infty) \to \Sigma \) which starts at \( A = \sigma(0, 0) \), and makes a 45\(^\circ\) degree angle with every meridian it meets. \([\text{Hint: represent } \gamma \text{ in the surface patch } \sigma \text{ by setting } \gamma(t) = \sigma(t, \Theta(t))].\]

PROBLEM 2

Consider a saddle surface \( S \) with surface patch \( \sigma : \mathbb{R} \to \mathbb{R}^3 \) given by

\[
\sigma(u, v) = \begin{pmatrix}
u \\
v \\
vuv
\end{pmatrix}.
\]

(1) Compute the first fundamental form of \( \sigma \).

(2) Find the area of the portion \( R = \{\sigma(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\} \) of the surface parametrized by \( \sigma \). (You may leave a double integral in the answer.)

(3) Compute the normal curvature \( \kappa_n \) and geodesic curvature \( \kappa_g \) of the curve \( \gamma \) on \( S \) given by \( \gamma(t) = \sigma(t, at) \) where \( a \in \mathbb{R} \) is a constant.

(4) Compute second fundamental form, the principal curvatures, and principal curvature directions of \( S \) at the point \( (1, 1, 1) \).