MATH 725, Spring 2000
Course Outline

1 Functional Analysis

The material for this section will mainly be taken from Rudin’s [1, 2]. Additional examples will be handed out as problem sheets.

**Hilbert spaces.** Definition, Orthonormal bases, Riesz representation theorem $H^* \cong H$. $\ell^2(\mathbb{N})$ and $L^2(\Omega)$.

**Banach spaces.** Definition. A list of examples of function spaces which are Banach spaces: $L^p(\Omega)$, $C^0(\Omega)$, $C^{0,\alpha}(\Omega)$, $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $W^{-m,p}(\Omega)$, $BV(\Omega)$. A function space which is not a Banach space: $S(\mathbb{R}^n)$.

Banach spaces which are (not necessarily) function spaces: the dual $X^*$, the space of bounded operators $\mathcal{L}(X,Y)$ with operator norm.

**Bounded operators.** Definition, some examples: differential operators $P(D)$: $C^m \to C^0$, integral operators of various types (Young’s inequality, infinite matrices on $\ell^p$, Hilbert-Schmidt operators).

The (Banach-) contraction mapping principle, “Neumann-series” (a.k.a. the geometric series for $(I + K)^{-1}$).

**General facts.** Hahn-Banach, Open Mapping and Closed Graph theorems. Closed subspaces in general do not have closed complements. Adjoint operators, $\text{ker} T^* = (\text{range} T)^\perp$, $\text{ker} T = ^\perp (\text{range} T)$.

**Weak convergence.** Definition, Banach-Alaoglu theorem. Riemann-Lebesgue lemma.

Compact operators and Fredholm operators.

2 Fourier series and integral

We still follow [1, 2].

**Fourier series on $\mathbb{T}^n$.** Uniform convergence of partial sums $s_N(f)$ if $f$ is Hölder continuous, convergence in $L^2(\mathbb{T}^n)$ for $f \in L^2$.

**Fourier integral.** Inversion formula for $f \in C^1_c(\mathbb{R}^n)$, pointwise for $f$ with $f, \hat{f} \in L^1(\mathbb{R}^n)$ and in the sense of distributions for $f \in S'(\mathbb{R}^n)$. 
The **Plancherel formula** and boundedness of the Fourier transform on $L^2(\mathbb{R}^n)$.

**Convolution and Fourier multipliers.** Boundedness on $L^2$, Hölder spaces (example of dyadic decomposition – notes will be provided) and without proof $L^p$. Elliptic estimates for constant coefficient operators.

## 3 Sobolevology

I will follow chapter 5 of Evans’ PDE book [3].

**Definition** of the spaces $W^{m,p}(\Omega)$ with $m \in \mathbb{Z}$ and of $W^{m,p}_0(\Omega)$ for any open subset $\Omega \subset \mathbb{R}^n$. Invariance under diffeomorphisms (coordinate changes) of $\Omega$.

**Trace theorems.** Restriction of $f \in W^{m,p}(\Omega)$ to a smooth submanifold makes sense for suitable values of $m$ and $p$. In particular one can restrict any $f \in W^{1,p}(\Omega)$ to $\partial \Omega$ if $\partial \Omega$ is smooth, and if $p > n$ then one can “restrict to a point,” i.e. any $f \in f \in W^{1,p}(\Omega)$ is defined everywhere. In fact $W^{1,p}(\Omega) \subset C^{0,1-n/p}(\Omega)$.

**Sobolev inequalities.** $\|f\|_{L^{n/(n-1)}} \leq C\|\nabla f\|_{L^1}$, and consequences for other values of $p$ than $p = 1$. Relation between the Sobolev inequality and the isoperimetric inequality.

**Compactness issues.** The Rellich-Kondrachov compactness theorem (and Arzela-Ascoli of course).

**Sobolev-like spaces.** (if time permits) BV(\Omega) and sets with finite perimeter. Quick solution to the problem of minimal surfaces.

**Solving PDEs by Hilbert-Sobolev space methods.** Solution of the Dirichlet problem by minimizing the Dirichlet integral. Solution of Poisson’s equation by using the Riesz representation theorem. A proof of the Riemann Mapping Theorem from complex analysis, using real variable methods (if time permits).

## References

