1. The 1D linear wave equation – weak solutions

We have considered the initial value problem for the wave equation in one space dimension:

\[(1a) \quad u_{tt} - u_{xx} = f(x,t)\]
\[(1b) \quad u(x,0) = g(x),\]
\[(1c) \quad u_t(x,0) = h(x)\]

We considered solutions of this equation on the domain \(\bar{U}\), where

\[U = \mathbb{R} \times (0,\infty), \quad \bar{U} = \mathbb{R} \times [0,\infty).\]

1.1. Definition. A function \(u\) is called a classical solution to (1) if \(u \in C^2(\bar{U})\), and if \(u\) satisfies (1).

1.2. Definition. A function \(u \in C^0(\bar{U})\) is a weak solution to (1) if for every \(\varphi \in C^\infty_c(\bar{U})\) one has

\[(2) \quad \iint_U \varphi f dx dt + \int_0^\infty \left\{ \varphi(x,0)h(x) - \varphi_t(x,0)g(x) \right\} dx = \iint_U (\varphi_{tt} - \varphi_{xx})u dx dt.\]

1.3. Theorem (consistency of classical and weak). Every classical solution is a weak solution.

(The proof: start with \(\iint_U \varphi f dx dt = \iint (u_{tt} - u_{xx})\varphi dx dt\) and integrate by parts.)

1.4. Theorem (explicit formula for all weak solutions). If \(u\) is a weak solution of (1) then it is given by

\[(3) \quad u(x,t) = \frac{1}{2} \{g(x+t) + g(x-t)\} + \frac{1}{2} \int_{x-t}^{x+t} h(x') dx' + \frac{1}{2} \iint_{T(x,t)} f(x',t') dx' dt',\]

where \(T(x,t)\) is the triangle with vertices \((x,t), (x-t,0)\) and \((x+t,0)\).

About the proof: If \(u\) is a classical solution then you can derive (3) by integrating \(u_{tt} - u_{xx}\) over the triangle \(T(x,t)\) and applying Green’s theorem.

To prove the general case choose a point \((\bar{x},\bar{t})\) \(\in U\) and an appropriate test function \(\varphi_\varepsilon\)

\[\varphi_\varepsilon(x,t) = \psi\left(\frac{x-t - \bar{x} + \bar{t}}{\varepsilon}\right) \cdot \psi\left(\frac{\bar{x} + \bar{t} - x - t}{\varepsilon}\right)\]

where \(\psi : \mathbb{R} \to \mathbb{R}\) is a smooth \((C^2)\) function which is nondecreasing, and which satisfies \(\psi(s) = 1\) for \(s \geq 1\), \(\psi(s) = 0\) for \(s \leq 0\).

Let \(\varepsilon \downarrow 0\) and carefully compute the limits of all the integrals you have in (2). The result is (3).
1.5. **Theorem (about convergence of weak solutions).** Let \( u_n \) be a sequence of weak solutions with data \((f_n, g_n, h_n)\). Assume that \((f_n, g_n, h_n, u_n)\) converge uniformly on compact sets to \((f, g, h, u)\). Then \( u \) is also a weak solution with data \((f, g, h)\).

This theorem is one reason to introduce weak solutions: note that one doesn’t assume that the derivatives of the solutions converge, just the solutions \( u_n \) themselves.

1.6. **Theorem (about existence of weak solutions).** If \( f, g, h \) are continuous functions then the function \( u \) defined by (3) is a weak solution of (1).

About the proof: When \( f \in C^1, h \in C^1 \) and \( g \in C^2 \) then you can check that \( u \) is in fact a classical solution to (1). To prove the general case where \((f, g, h)\) are merely continuous, we used an approximation argument.

2. **Problems**

Use the definitions and theorems above to solve these problems:

2.1. **Homogeneous wave equation.** Assume \( f = 0 \).

(a) Show that every weak solution \( u \) can be written as

\[
  u(x, t) = \alpha(x - t) + \beta(x + t).
\]

for certain functions \( \alpha, \beta : \mathbb{R} \rightarrow \mathbb{R} \).

(b) Relate the functions \( \alpha, \beta \) to the initial values \( g, h \).

(c) Suppose \( \alpha, \beta \in C^0(\mathbb{R}) \) are arbitrary continuous functions; is \( u(x, t) = \alpha(x - t) + \beta(x + t) \) then necessarily a weak solution according to our definition 1.2?

2.2. **An example of weak solutions.** The function \( u(x, t) = |x| \) satisfies the wave equation \( u_{tt} - u_{xx} = 0 \) for all \((x, t)\) except \( x = 0, t > 0 \). It also satisfies \( u(x, 0) = |x| \) and \( u_t(x, 0) = 0 \) for all \( x \).

(a) Find the weak solution to (1) with \( f = 0, g(x) = |x| \) and \( h(x) = 0 \).

(b) Is \( u \) a weak solution to the wave equation?

2.3. **Another example.** The function

\[
  u(x, t) = \begin{cases} 
  x^2 + t^2 & x \geq 0, \\
  0 & x < 0
  \end{cases}
\]

satisfies the homogeneous wave equation for all \((x, t)\) except \( x = 0 \) again.

Find the weak solution of \( u_{tt} - u_{xx} = 0 \) with \( u(x, 0) = x^2 \) for \( x > 0 \), \( u(x, 0) = 0 \) for \( x < 0 \) and \( u_t(x, 0) = 0 \) for all \( x \).

2.4. **An example involving convergence.** Consider \( u_n(x, t) = \frac{1}{n} \cos(nx) \cos(nt) \).

(a) Show that each \( u_n \) is a classical solution of (1) with \( f = 0, h = 0 \) and \( g_n = \frac{1}{n} \cos nx \).

(b) Show that the \( u_n \) converge uniformly, and verify that the limit is again a solution.

(c) Do the partial derivatives of first and second order converge as \( n \to \infty \)?
3. Semilinear equations

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and consider the semilinear wave equation

(4) \quad u_{tt} - u_{xx} = f(u), \quad u(x,0) = g(x) \text{ and } u_t(x,0) = h(x).

Examples are the Klein-Gordon equation

$$u_{tt} - u_{xx} = m^2 u,$$

with $m > 0$ a constant, and the Sine-Gordon equation

$$u_{tt} - u_{xx} = \sin u.$$

3.1. Definition. A function $u \in C^0(\bar{U})$ is a weak solution to (4) if $u$ is a weak solution to the equation

$$u_{tt} - u_{xx} = F(x,t), \quad u(x,0) = g(x) \text{ and } u_t(x,0) = h(x).$$

where $F(x,t) = f(u(x,t)).$

3.2. Theorem. (Existence) If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous then (4) has a unique weak solution $u \in C^0(\bar{U})$ for any initial data $g,h \in C^0(\mathbb{R}).$

The proof proceeds by applying Picard-iteration to the integral equation

$$u(x,t) = u_0(x,t) + \int_0^t \int_{\Gamma(x,t)} f(u(x',t'))dx'dt'.$$

where

$$u_0 \overset{\text{def}}{=} \frac{1}{2} \{ g(x + t) + g(x - t) \} + \frac{1}{2} \int_{x-t}^{x+t} h(x')dx'.$$

3.3. Theorem. (Local Existence) Let $g,h \in C^0(\mathbb{R})$ be given. If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then there exist a $T > 0$, and a $u \in C^0(\mathbb{R} \times [0,T])$ such that $u$ is a weak solution to (4) on $\mathbb{R} \times [0,T)$.

This theorem follows by applying the previous theorem to a modified $f$. Let

$$f_M(u) = \begin{cases} f(u) & \text{for } |u| \leq M, \\ f(M) & \text{for } u > M \text{ and} \\ f(-M) & \text{for } u < -M. \end{cases}$$

The $f_M$ is Lipschitz continuous and the initial value problem

$$u_{tt} - u_{xx} = f_M(u), \quad u(x,0) = g(x) \text{ and } u_t(x,0) = h(x).$$

has a unique solution $u_M \in C^0(\bar{U}).$

If $M > \sup |g(x)|$ then there exists $T > 0$ such that $|u_M(x,t)| < M$ for all $t \in [0,T)$. Hence $u_M$ is a weak solution of the original equation (4) for $(x,t) \in \mathbb{R} \times [0,T)$.

3.4. Theorem. (Uniqueness) Let $u_1,u_2$ be weak solutions on $\mathbb{R} \times [0,T)$ of (4) with initial data $(g_1,h_1)$ and $(g_2,h_2)$ respectively. If $g_1(x) = g_2(x)$ and $h_1(x) = h_2(x)$ on an interval $[\bar{x} - \bar{t}, \bar{x} + \bar{t}]$, then $u_1 = u_2$ on the triangle

$$\{(x,t) : 0 \leq t \leq \bar{t}, |x-\bar{x}| < \bar{t} - t\}.$$
4. Regularity

Let \( u \) be the weak solution of

\[
\Box u(x,t) = f(x,t), \quad u(x,0) = g(x), u_t(x,0) = h(x).
\]

so that \( u \) is given by (4).

If \( u \) were a smooth solution, and if \( f, g, h \) were differentiable functions, then you could differentiate the PDE w.r.t. \( x \) or \( t \). You would find that \( v = u_x \) and \( w = u_t \) satisfy

(5) \[ \Box v = f_x, \text{ with initial conditions } v = g_x, \ u_t = h_x \]

and

(6) \[ \Box w = f_t, \text{ with initial conditions } w = h, \ w_t = g_x + f(x,0) \]

If the data \( (f, g, h) \) are merely continuous then the solution \( u \) need not be differentiable. In general it can happen that the solution is differentiable \((u \in C^1)\), but that the derivatives \( v = u_x \) and \( w = u_t \) are not weak solutions of the above equations (6) and (5).

The explicit solution (3) implies that for arbitrary continuous \( f, g, h \) one has a weak solution which is continuous. By differentiating the integrals in (5) one can prove that to get a solution \( u \in C^1 \) (i.e. one for which \( u_x \) and \( u_t \) are continuous) you only have to assume that \( g \in C^1(\mathbb{R}) \):

4.1. Theorem. If \( u \in C^0(\overline{U}) \) is the weak solution to (1), then

(7) \[ f, h \in C^0, g \in C^1 \implies u \in C^1 \]

Proof: you have

\[
 u(x,t) = \frac{1}{2} \{g(x+t) + g(x-t)\} + \frac{1}{2} \int_{x-t}^{x+t} h(x')dx' + \frac{1}{2} \int_0^{t} \int_{x-t}^{x+t} f(x',t') \ dx' \ dt'.
\]

If \( f, h \in C^0, g \in C^1 \) then you can differentiate under the integral and conclude that \( u_x, u_t \) are continuous on \( \overline{U} \).

This Theorem does not imply that (5) and (6) hold. To prove that it turns out that you also need \( f_x, f_t, h_x, g_x \) and \( g_{xx} \) to exist and to be continuous.

(8) \[ h \in C^1, f \in C^1, g \in C^2 \implies u \in C^1 \text{ and } u_t \text{ and } u_x \text{ are weak solutions of (5) and (6)}. \]

4.2. Proof that \( u_x \) satisfies (5). There are at least two proofs: you can use the integral form (3) of the solution, or you can directly use the definition 1.2 of weak solution. To prove that \( u_x \) satisfies (5) we must show that for every test function \( \varphi \in C^\infty_c(U) \) one has

(9) \[
\iint_{U} \Box \varphi \ u_x \ dxdt + \int_{\partial U} \{\varphi (g_x - \varphi h_x)\} dx = \iint_{U} \varphi f_x \ dxdt.
\]

Integrate by parts twice in the first double integral:

\[
\int_0^\infty \int_\mathbb{R} \Box \varphi \ u_x \ dxdt = - \int_0^\infty \int_\mathbb{R} \varphi \ u_x \ dxdt.
\]
If \( \varphi \) is a test function then so is \( \varphi_x \), so, since \( u \) is a weak solution we get

\[
- \int_0^\infty \int \Box \varphi_x \ u \ dx \ dt = \int \{ \varphi_t g - \varphi_x h \} \ dx - \int \varphi_x f \ dx \ dt.
\]

Integrate by parts in the \( x \) direction again, and you find

\[
- \int_0^\infty \int \Box \varphi_x \ u \ dx \ dt = - \int \{ \varphi_t g_x - \varphi h_x \} \ dx + \int \varphi f_x \ dx \ dt.
\]

This implies (9).

5. Problems

5.1. **An example.** Give an example of a solution of

\[
\Box u = 0, \quad u(x, 0) = g(x), \quad u_t(x, 0) = h(x)
\]

for which

- \( u \in C^1(\overline{U}) \)
- \( v = u_x \) is not a weak solution of

\[
\Box v = 0, \quad v(x, 0) = \tilde{g}(x), \quad v_t(x, 0) = \tilde{h}(x)
\]

for any \( \tilde{g}, \tilde{h} \in C^0(\mathbb{R}) \).

Can you find such a solution \( u \) if \( h(x) = 0 \)?

5.2. **Proof that** \( u_t \) **satisfies** (6). Above in §4.2 we proved half of (8). Prove the other half, i.e. prove directly from the definition of weak solution that \( u_t \) satisfies (6) under the hypotheses in (8).

5.3. **Higher regularity.** Show by induction on \( n \geq 1 \) that if \( u \) is a weak solution of

\[
\Box u = f(x, t), \quad u(x, 0) = g(x), \quad u_t(x, 0) = h(x)
\]

then

\[
f \in C^{n-1}(\overline{U}), \quad g \in C^n(\mathbb{R}), \quad h \in C^{n-1}(\mathbb{R}) \quad \implies \quad u \in C^n(\overline{U}).
\]

5.4. **Regularity for semilinear equations.** Let \( n \geq 1 \). Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a \( C^{n-1} \) function, and let \( u \in C^n(\mathbb{R} \times [0, T)) \) be a maximal weak solution of

\[
\Box u = f(u), \quad u(x, 0) = g(x), \quad u_t(x, 0) = h(x)
\]

with \( g \in C^n(\mathbb{R}) \) and \( h \in C^{n-1}(\mathbb{R}) \). Show that \( u \in C^n(\mathbb{R} \times [0, T)) \). In other words, show that the solution is as smooth as the initial data. (Hint: use the previous problem and proceed with induction on \( n \).)