Problem 1 (3 points)
Find the point on the surface \( z = xy + 1 \) nearest the origin.

Solution:

Let 
\[
f(x, y, z) = x^2 + y^2 + z^2.
\]
That is, \( f \) is the square of the distance from point \((x, y, z)\) to the origin. We want to minimize \( f \).

Let 
\[
g(x, y, z) = z - xy - 1.
\]

We must solve

\[
\nabla f = \lambda \nabla g
\]

and

\[
g = 0.
\]

By equation (1) we have that

\[
\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix}.
\]

So we must solve the system of equations

\[
2x = -\lambda y \tag{3}
\]

\[
2y = -\lambda x \tag{4}
\]

\[
2z = \lambda \tag{5}
\]

By equation (4) we have that

\[
y = -\frac{\lambda x}{2}.
\]

Substituing this into equation (3) we have that

\[
4x = \lambda^2 x.
\]

Thus, either \( x = 0 \) or \( 4 = \lambda^2 \).

Case 1: \( x = 0 \)

If \( x = 0 \) then \( y = 0 \). Also, since the point \((x, y, z)\) must satisfy \( g = 0 \), we have that

\[
z - 0 - 1 = 0.
\]

That is, \( z = 1 \). Our candidate to minimum in this case, then, is point \((0, 0, 1)\).

Case 2: \( 4 = \lambda^2 \)

If \( 4 = \lambda^2 \) then either \( \lambda = 2 \) or \( \lambda = -2 \).
If \( \lambda = 2 \) then by equation \((3)\), we have that \( x = -y \). Also, by equation \((5)\), \( z = 1 \).

The point \((x, -x, 1)\) must satisfy \( g = 0 \). Thus,

\[
1 + x^2 - 1 = 0
\]

That is, \( x = 0 \). As in the previous case, we get the point \((0, 0, 1)\).

If \( \lambda = -2 \) then by equation \((3)\), we have that \( x = y \). Also, by equation \((5)\), \( z = -1 \).

The point \((x, x, -1)\) must satisfy \( g = 0 \). Thus,

\[
-1 - x^2 - 1 = 0
\]

That is, \( x^2 = -2 \). This can not be true, so we can disregard point \((x, x, -1)\).

We claim that point \((0, 0, 1)\) is in fact a minimum (and not a maximum) of \( f \). Notice that \( f(0, 0, 1) = 1 \). It suffices then to find some point in the given surface such that it’s distance to the origin is larger than 1 to know that \((0, 0, 1)\) is in fact a minimum and not a maximum. Consider, for example, point \((-1, 1, 0)\). Notice that this point satisfies \( g = 0 \). But \( f(-1, 1, 0) = 2 > f(0, 0, 1) \). Thus, \( f \) is not a maximum so it must be the minimum.

That is, the point on the surface \( z = xy + 1 \) closest to the origin is \((0, 0, 1)\).