MATH 845: HOMEWORK 6, DUE MAY 4.

11. Every year, the London Sunday Telegraph has a New Year’s Quiz. In 1995, two of the questions were the following:
   (a) Solve the equation \( \frac{A^3}{B^3} + \frac{C^3}{D^3} = 6 \), where \( A, B, C, D \) are all positive whole numbers below 100.
   (b) (A special question with a 450 pound prize.) Either give a second solution to the above equation where the four variables are all whole numbers above 100 \((A, B \text{ and } C, D \text{ relatively prime})\), or demonstrate that no such second solution can exist.
   [It’s too late to claim the prize.]

12. The integer equation \( a^4 + ma^2b^2 + b^4 = c^2 \) \((*)\) \((a, b) = 1, a, b > 0 \) was studied by Fermat and Euler. A solution is called trivial if either \( ab = 0 \) or \( a = b = 1 \).
   (a) Let \( E \) be the elliptic curve over \( \mathbb{Q} \) given by \( y^2 = x^3 + mx^2 + x \). Show that \((*)\) has a nontrivial solution if and only if the Mordell-Weil rank of \( E \) is nonzero.
   (b) Euler showed that for \( m = 14 \), there are only trivial solutions of \((*)\). Prove this.
   (c) Suppose \( L(E, s) = \sum c_n/n^s \). Since \( E \) is modular (why?), work of Buhler, Gross, et al. gives the formula:
   \[
   L(E, 1) = \sum c_n(\exp(-2\pi nx/\sqrt{N}) + \epsilon \exp(-2\pi n/(x\sqrt{N}))) / n
   \]
   where \( x \) is any positive real number, \( N \) is the conductor of \( E \), and \( \epsilon = \pm 1 \) its root number.
   Explain why this formula gives a means of computing \( \epsilon \). In the case \( \epsilon = 1 \), obtain a simpler formula for \( L(E, 1) \).
   (d) For \( m = 145 \), Euler claimed that \((*)\) had a nontrivial solution, namely \((159, 40)\). Show that he was mistaken.
   (e) Kolyvagin proved the weak Birch Swinnerton-Dyer conjecture for modular elliptic curves over \( \mathbb{Q} \) whose L-functions vanish to order at most 1 at \( s = 1 \). Show how this gives a way to prove that for a given \( m \) there are no nontrivial solutions. For \( m = 145 \) compute the coefficients of \( L(E, s) \) up to \( n = 10 \) (the conductor of \( E \) is 48048 and root number 1) - using (c), is this enough to determine whether \((*)\) has nontrivial solutions for \( m = 145 \)?