MATH 240; FINAL EXAM, 150 points, 20 December, 2005 (R.A.Brualdi)

TOTAL SCORE (10 problems; plus one 15 point Bonus):

Name:

TA: Anders Hendrickson (circle time)   Mon 12:05 Mon 1:20 Wed 12:05 Wed. 1:20

Do NOT compute factorials or binomial coefficients.

1. [10 points] Let \( A \) be a set of 7 elements and let \( B \) be a set of 9 elements. What are:
   
   • The number of \textbf{injective} (one-to-one) functions \( f \) from \( A \) to \( B \):
   
   • The number of \textbf{binary relations} \( R \) from \( A \) to \( B \):

2. [10 points] Let \( A = \{a, b, c\} \). Draw the \textbf{diagram} of the partially ordered \((S, \subseteq)\) where \( S \) is the power set of \( A \). Then \textbf{topologically sort} the elements of this partially ordered set.
3. [15 points] Twenty-four people are to be transported by 6 cars (a Toyota, a Subaru, a Ford, a Jeep, a Chevrolet, and a Chrysler) with 4 people per car.

- In how many ways can the transportation be arranged?

- If one person in each car is to have a designated driver, how many ways can the transportation be arranged?

4. [10 points] For \( n \geq 1 \), Let \( h_n \) denote the number of ways for a person to climb a flight of \( n \) stairs when the person takes 1, 3, or 4 steps at a time. What is a recurrence relation for \( h_n \) with initial conditions?
5. [15 points] The transitive closure of a binary relation $R$ on a set of 7 elements \{1, 2, 3, 4, 5, 6, 7\} is computed by Warshall’s algorithm producing matrices $W_0, W_1, W_2, W_3, W_4, W_5, W_6, W_7$.

- Give a formula for computing the entries $w_{ij}^{[4]}$ of $W_4$ from the entries $w_{ij}^{[3]}$ of $W_3$:

- If (unspecified entries are 0)

\[
W_3 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

, determine $W_4$ (use the same matrix diagram).
6. [30 points] Consider the binary relations $R$ on a set $A$ defined below.

- What are the three defining properties for $R$ to be an equivalence relation?

- What are the three defining properties for $R$ to be a partial order?

- Let $R$ be the relation on $A = \{-3, -2, -1, 0, 1, 2, 3\}$ defined by $aRb$ provided $|a| = |b|$ (absolute value). Is $R$ an equivalence relation, total order, partial order, or none of these? (circle one)
If an equivalence relation, how many different equivalence classes are there and what are they?

- Let $R$ be the relation on $A = \{1, 2, 3, \ldots, 20\}$ defined by $aRb$ provided that $a$ is a divisor of $b$. Is $R$ an equivalence relation, total order, partial order, or none of these? (circle one).
If an equivalence relation, how many different equivalence classes are there and what are they?
• Let $R$ be the relation on the set $A = \{1, 2, 3, \ldots, 19, 20\}$ of 20 elements defined by $aRb$ provided that $a \equiv b \pmod{6}$. Is $R$ an equivalence relation, total order, partial order, or none of these? (circle one)

If an equivalence relation, how many different equivalence classes are there and what are they?

7. [10 points] Let $A$ be a set of 4 elements, and let $R$ be a binary relation on $A$.

• If $R$ is a symmetric relation and $M_R$ has 0’s and 1’s as shown:

$$
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
$$

then show what the other entries of $M_R$ are.

• Let $M_R$ have 0’s and 1’s in positions as shown:

$$
\begin{bmatrix}
1 & 1 & \ & \ \\
0 & 0 & \ & \ \\
\ & \ & 1 & 0 \\
\ & \ & 1 & 1
\end{bmatrix}
$$

The number of ways to complete this to an anti-symmetric, binary relation on $A$ equals:
8. [10 points] Consider the $n$-cube graphs $Q_n$:

- What are the degrees of the vertices of $Q_n$?

- Calculate the number of edges of $Q_n$. 


9. [20 points] Consider the poset \((S, \leq)\) whose diagram is given below.

Determine

- All \textbf{maximal} elements.

- All \textbf{minimal} elements.

- All the \textbf{upper bounds} of \(a\) and \(b\), and then the \textbf{LUB} of \(a\) and \(b\) if it exists.

- All the \textbf{lower bounds} of \(a\) and \(b\), and then the \textbf{GLB} of \(a\) and \(b\) if it exists.
10. [20 points]

• The following is the postfix (postorder) form of a logical expression. What is its (unambiguous) usual form?

\[ p \ q \ r \ \lor \ q \ p \ \land \ p \ r \ \land \ \lor \]

• What is the prefix (preorder) form of the algebraic expression

\[ ((x \times y) + z) - ((z \times (u + v)) + ((y \times z) \times (x + y))) \]
Bonus Problem (3+2+2+8=15 points)

• Give a recursive definition of a Full Binary Tree (FBT).

• Give a recursive definition of the height $h(T)$ of a FBT $T$.

• Give a recursive definition of the number $n(T)$ of vertices of a FBT $T$. 
• Prove by structural induction that $n(T) \leq 2^{h(T)+1} - 1$ for a FBT $T$. 