1. (12 points; 4 points each) Let $A$ be an $m \times n$ matrix with rows $\alpha_1, \alpha_2, \ldots, \alpha_m$ and let $B$ be a matrix with columns $\beta_1, \beta_2, \ldots, \beta_n$. Then,

- The **entry** of $AB$ in position $(i, j)$ is:
  $$\alpha_i \cdot \beta_j$$

- The **rows** of $AB$ are:
  $$\alpha_1 \cdot B, \ldots, \alpha_m \cdot B$$

- The **columns** of $AB$ are:
  $$A\beta_1, \ldots, A\beta_n$$

2. (8 points) The augmented matrix of a system $Ax = b$ of 4 equations in 5 unknowns $x_1, x_2, x_3, x_4, x_5$ has rref (reduced row echelon form) equal to

$$
\begin{bmatrix}
1 & -1 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 3 & -4 \\
0 & 0 & 0 & 1 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Is the system $Ax = b$ **consistent**? If so, what is the **solution set**?

The systems is consistent. The solution set is given by:

- $x_1 = 2 + x_2 - x_5$
- $x_2 = r$ arbitrary
- $x_3 = -4 - 3x_5$
- $x_4 = 6 - 5x_5$
- $x_5 = s$ arbitrary

3. (8 points; 4 points each) Let $A$ be a matrix of size 3 by 4.

1. What **elementary matrix** $E$ has the property that $AE$ is obtained from $A$ by adding 3 times column 2 to column 4?

$$E = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$
2. What is the inverse of $E$?

$$E^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

4. (8 points) Let $Ax = b$ be a nonhomogeneous system of $m$ equations in $n$ unknowns, and let $x = u$ be a solution of this system. Let $x = v$ be a solution of the homogeneous system $Ax = 0$ with the same coefficient matrix $A$. **Prove that, for every scalar $k$, $u + kv$ is a solution of the homogeneous system $Ax = b$.**

We have $A(u + kv) = Au + A(kv) = Au + k(Av) = b + k(0) = b + 0 = b$.

5. (40 points; 4 points each) Given that $A$ and $B$ are nonsingular matrices of the same order with inverses $A^{-1}$ and $B^{-1}$, respectively, with determinants $\det A = 3$ and $\det B = 5$. Either give the answer to each of the following questions or say that with the given information, it is not possible to say what the answer is (abbreviate: not possible):

(a) The inverse of $AB$ is: $B^{-1}A^{-1}$

(b) The inverse of $A + B$ is: NP

(c) The inverse of $A^T$ is: $(A^{-1})^T$

(d) The solution set of $Ax = 0$ is: only the trivial solution $x = 0$.

(e) The solution set of $Ax = b$ is: only $A^{-1}b$

(f) The inverse of $A^2B^3$ is: $(B^{-1})^3(A^{-1})^2$

(g) The rref of $A$ is: $I_n$

(h) The determinant of $AB^{-1}$ is: $\det(A)/\det(B) = 3/5$.

(i) The determinant of $A + B$ is: NP

(j) The inverse of the adjoint of $A$ is: $\frac{1}{3}A$

6. (8 points) Let $A$ be a nonsingular matrix of order $n$. **Prove that the inverse of $A$ is unique**; that is, prove that if $B$ and $C$ are matrices such that

$$AB = BA = I_n \text{ and } AC = CA = I_n,$$

then $B = C$.

**NOTE:** One cannot use $A^{-1}$ for that it tantamount to assuming that the inverse of $A$ is unique.
So we have:

\[ B = B I_n = B(AC) = (BA)C = I_n C = C \]

7. (8 points) Evaluate the determinant of the following matrix

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 0 \\
3 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 \\
0 & 0 & 6 & 0 & 7 \\
0 & 8 & 0 & 0 & 9
\end{bmatrix}
\]

Using the definition of the determinant, we see that there are only 2 nonzero terms:

1 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \text{ corresponding to the odd permutation } 1, 3, 4, 5, 2 \text{ and }
2 \cdot 3 \cdot 5 \cdot 6 \cdot 9 \text{ corresponding to the even permutation } 2, 1, 4, 3, 5

So determinant is \(-1120 + 1620 = 500\)

8. (8 points) A matrix \(A\) of order 5 has \(\det A = 10\) and the submatrix \(M_{2,5}\) of \(A\) obtained by deleting row 2 and column 5 has \(\det M_{2,5} = 3\). Which entry of \(A^{-1}\) is now determined and what is the value of that entry?

The entry in position (5,2) and it is

\[
(-1)^{2+5} \frac{3}{10} = -\frac{3}{10}
\]