1. Let \( A = (A_1, A_2, \ldots, A_n) \) be a family of subsets of a finite set \( E \) and let \( m \geq 1 \). Prove that \( A \) has \( m \) pairwise disjoint transversals iff

\[
|A(K)| \geq m|K| \quad (K \subseteq \{1, 2, \ldots, n\}).
\]

2. Let \( A = (A_1, A_2, \ldots, A_n) \) be a family of subsets of a finite set \( E \) and let \( m \geq 1 \). Prove that \( A \) has a SR in which no element occurs more than \( m \) times if and only if

\[
m|A(K)| \geq |K| \quad (K \subseteq \{1, 2, \ldots, n\}).
\]

3. Let \( a_1, a_2, \ldots, a_{n^2+1} \) be a permutation of \( \{1, 2, \ldots, n^2+1\} \). Use Dilworth’s Theorem to show that \( a_1, a_2, \ldots, a_{n^2+1} \) has a monotone sequence of length \( n+1 \). (Note: there is a nice proof of this using the pigeon-hole principle; if you have never seen it you should try that too.)

4. Use Rado’s theorem on the existence of an independent transversal and the rank function of a transversal matroid to show the following: If \( A = (A_1, A_2, \ldots, A_n) \) and \( B = (B_1, B_2, \ldots, B_n) \) are two families of subsets of \( E \), then \( A \) and \( B \) have a common transversal (i.e. a set that is a transversal of both families) if and only if

\[
|A(K) \cap B(L)| \geq |K| + |L| - n \quad (K, L \subseteq \{1, 2, \ldots, n\}).
\]

5. (Another way to compute the Möbius function of a poset \( P : X, \preceq \).) Let \( \hat{0} \) and \( \hat{1} \) be two new elements and extend \( P \) to \( \hat{P} : \hat{X}, \preceq \) by defining \( \hat{0} < x < \hat{1} \) for all \( x \in X \). (Of course, if \( P \) is already a lattice, one doesn’t have to do the extension.)

Prove that

\[
\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots
\]

where \( c_i \) equals the number of chains (not necessarily saturated) of the form \( \hat{0} = x_0 < x_1 < \cdots < x_i = \hat{1} \).

HINT: Consider \( \zeta = 1 + (1 - \zeta) \).

6. Let \( n \) be a positive integer, let \( q \) be a prime power and let \( F \) be a field of \( q \) elements. The subspaces of \( V = F^n \) ordered by inclusion determine a finite lattice \( P_n \).

(a) Prove that that the Möbius function for this lattice satisfies

\[
\mu(0, 1) = (-1)^n q^{\binom{n}{2}},
\]

from which one obtains the complete Möbius function since every interval is isomorphic to \( P_m \) for some \( m \leq n \).

HINT: Use our Lemma: If \( L \) is a lattice and \( a \in L \) with \( a > 0 \), then

\[
\mu(0, 1) = - \sum_{1 \neq x : x \lor a = 1} \mu(0, x)
\]

by choosing \( a \) to be a 1-dimensional space and considering what \( x \) satisfy \( x \lor a = 1 \).
(b) Use Möbius inversion to show that the number of spanning sets of vectors of $V$ is

$$
\sum_{k=0}^{n} \binom{n}{k} q^{\binom{n-k}{2}} (2q^k - 1).
$$

where

$$
\binom{n}{k} = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}.
$$

7. Let $A = (A_1, A_2, \ldots, A_n)$ be a family of subsets of a set $E$ on which there is defined a matroid with rank function $r(\cdot)$. Prove that $A$ has an independent partial transversal of size $t$ if and only if

$$
r(A(K)) \geq |K| - (n - t) \quad (K \subseteq \{1, 2, \ldots, n\}).
$$

HINT: See the outline I gave in class on October 3, 2006

8. Let $\Pi$ be a system of points and lines (and an incidence relation) such that $\Pi$ has a total of $n^2 + n + 1$ points with exactly $n + 1$ points on a line, and exactly $n + 1$ lines through a point. Assume that two distinct lines intersect in exactly one point, and that there exist 4 points no 3 on a common line. Prove that $\Pi$ is a projective plane.

9. For each integer $n \geq 4$ find a partial latin square of order $n$ which cannot be embedded in a latin square of any order $m < 2n$.

10. Complete the following partial latin square of order 6 to a latin square of order 6:

$$
\begin{bmatrix}
1 & 2 & 3 & * & * & *\\
* & * & * & 4 & * & *\\
* & * & * & 5 & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{bmatrix}.
$$

11. Let $G = \{a_1, a_2, \ldots, a_n\}$ be an abelian group of odd order $n$, written additively. Let $A$ be the addition table for $G$ and let $B$ be the subtraction table for $G$. Prove that $A$ and $B$ are orthogonal latin squares of order $n$.

12. Let $\lambda$ and $\mu$ be two partitions of the same integer $t$. Prove that the Kostka number $K_{\lambda,\mu} \neq 0$ if and only if $\mu \preceq \lambda$. Conclude that the number of nonnegative integral matrices with row sum vector $R$ and column sum vector $S$ is given by:

$$
\kappa'(R, S) = \sum_{R, S \preceq \lambda} K_{\lambda, R} K_{\lambda, S}
$$