Rings, domains, and fields

1. Determine if the set $R = \{ a + b\sqrt[3]{3} | a, b \in \mathbb{Q} \}$ is a ring with respect to the usual operations of addition and multiplication. If so, is it also a field?

2. Characterize the units in $M_2(\mathbb{Z})$. Then list the units in $M_2(\mathbb{Z}_2)$.

3. Show that $R = \{ a + b\sqrt{3}i | a, b \in \mathbb{Z} \}$ is a ring.

The complex numbers

4. Prove the following properties of the modulus of a complex number. Let $z, w \in \mathbb{C}$.
   
   a. $|zw| = |z||w|
   
   b. $|\overline{z}| = |z|
   
   c. $|z|^2 = z\overline{z}$

5. Find the sixth roots of $-3i$. Express your answers in the (exact) form $z = a + bi$ without trigonometric functions, and then plot them in the complex plane.

Euclidean algorithm for polynomials

6. Apply the division algorithm to the polynomials $f(x), g(x) \in \mathbb{Z}_7[x]$, where

   \[ f(x) = x^6 + 3x^5 + 4x^2 - 3x + 2 \quad \text{and} \quad g(x) = x^2 + 2x - 3. \]

   Clearly identify $q(x)$ and $r(x)$.

7. Find the greatest common divisor $d(x)$ for the polynomials $f(x), g(x) \in \mathbb{C}[x]$, where

   \[ f(x) = x^2 + 1 \quad \text{and} \quad g(x) = x^2 - i + 2, \]

   and find $s(x), t(x) \in \mathbb{C}[x]$ to express $d(x) = s(x)f(x) + t(x)g(x)$.

I discussed these exercises with the following people: