Problem Set 6
Math 541, Fall 2014
Due: Thursday, October 23

Ring homomorphisms and ideals

1. Find all ring homomorphisms:
   a. \( \phi : \mathbb{Z}_2 \rightarrow \mathbb{Z} \)
   b. \( \phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_6 \)
   c. \( \phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_2 \)

2. Prove that if \( p \) is prime and \( \phi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p, \phi(a) = a^p \), is a ring homomorphism.

3. Find all ideals in \( \mathbb{Z} \) and in \( \mathbb{Z}_6 \).

4. Let \( R \) be a commutative ring with 1, and let \( a_1, \ldots, a_n \in R \). Show that

\[
\langle a_1, \ldots, a_n \rangle := \{ r_1a_1 + \cdots + r_na_n \mid r_i \in R \text{ for all } i \} \subseteq R
\]

is an ideal in \( R \).

5. Let \( R \) be a commutative ring with 1, and let \( I, J \subset R \) be ideals. Define

\[
I \cap J = \{ a \in R \mid a \in I \text{ and } a \in J \} \quad \text{and} \quad I + J = \{ a + b \in R \mid a \in I, b \in J \}.
\]

a. Prove that \( I \cap J \) and \( I + J \) are ideals.

b. Suppose \( R = \mathbb{Z} \) or \( F[x] \) for a field \( F \), \( I = \langle a \rangle \), and \( J = \langle b \rangle \). Identify \( I \cap J \) and \( I + J \) in terms of \( a \) and \( b \).

c. Let \( a_1, \ldots, a_n \in R \). Prove that \( \langle a_1, \ldots, a_n \rangle = \langle a_1 \rangle + \cdots + \langle a_n \rangle \).

6. Let \( R \) be a commutative ring with 1.

a. Prove that if \( I \subseteq R \) is an ideal and \( 1 \in I \), then \( I = R \).

b. Prove that \( a \in R \) is a unit if and only if \( \langle a \rangle = R \).

c. Prove that the only ideals in \( R \) are \( \langle 0 \rangle \) and \( R \) if and only if \( R \) is a field.

I discussed these exercises with the following people: