Quotient rings

1. Give the addition and multiplication tables of \( \mathbb{Z}_2[x]/\langle x^3 + x + 1 \rangle \).

2. Let \( R \) and \( S \) be commutative rings and \( \phi: R \to S \) be a ring homomorphism.
   a. Given an ideal \( J \subseteq S \), define \( \phi^{-1}(J) := \{ a \in R \mid \phi(a) \in J \} \subseteq R \). Prove that this is an ideal in \( R \).
   b. Given an ideal \( I \subseteq R \), define \( \phi(I) := \{ \phi(a) \mid a \in I \} \subseteq S \). Prove that \( \phi(I) \) is an ideal in \( S \), provided that \( \phi \) maps onto \( S \).

3. An element \( a \) of a commutative ring \( R \) is called nilpotent if \( a^n = 0 \) for some positive integer \( n \).
   a. Find the nilpotent elements in \( \mathbb{Z}_8 \).
   b. Find the nilpotent elements in \( \mathbb{Z}_2[x]/\langle x^3 \rangle \).
   c. Show that the collection \( N \) of all nilpotent elements in \( R \) is an ideal.
   d. Show that the quotient ring \( R/N \) has no nonzero nilpotent elements.

Ring isomorphisms

4. a. Prove that the function \( \phi: \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{2}] \) defined by \( \phi(a + b\sqrt{2}) = a - b\sqrt{2} \) is a ring isomorphism.
   b. Define the function \( \phi: \mathbb{Q}[\sqrt{3}] \to \mathbb{Q}[\sqrt{7}] \) by \( \phi(a + b\sqrt{3}) = a + b\sqrt{7} \). Is \( \phi \) a ring isomorphism? Is there any isomorphism between these rings?

5. Establish the following isomorphisms by using the Fundamental Isomorphism Theorem:
   a. \( \mathbb{R}[x]/\langle x^2 + 6 \rangle \cong \mathbb{C} \)
   b. \( \mathbb{Q}[x]/\langle x^2 + x + 1 \rangle \cong \mathbb{Q}[\sqrt{3}i] \)
   c. \( \mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12} \)

6. Let \( F \) be a field, \( f(x) \in F[x] \), and \( K \) be a field extension of \( F \) containing the root \( \alpha \) of \( f(x) \).
   a. If \( \sigma: K \to K \) is a ring isomorphism with the property that \( \sigma(a) = a \) for all \( a \in F \), show that \( \sigma(\alpha) \) is likewise a root of \( f(x) \).
   b. Apply (a) to show that the complex roots of a real polynomial occur in conjugate pairs.
   c. Apply (a) to show that if \( n \in \mathbb{N} \) is not a perfect square, and \( \sqrt{n} \) is a root of \( f(x) \in \mathbb{Q}[x] \), then \(-\sqrt{n} \) is a root as well.

I discussed these exercises with the following people: