Groups

1. Let $G$ be a group and fix $a \in G$. Prove that $C_a = \{ x \in G \mid ax = xa \}$ is itself a group, called the centralizer of $a$.

Cyclic groups

2. a. List all of the generators of $\mathbb{Z}_{20}$.
   b. List the elements of the subgroups $\langle 3 \rangle$ and $\langle 7 \rangle$ in $U(20) = \mathbb{Z}_{20}^\times$.
   c. Find all subgroups of $\mathbb{Z}_{18}$ and $U(11) = \mathbb{Z}_{11}^\times$.

3. a. Let $a$ be an element in a group. If $|a| = n$, show that $\langle a^k \rangle = \langle a^{\text{gcd}(n,k)} \rangle$.
   b. Let $a$ be an element in a group. Suppose that $|a| = 24$. Find a generator of $\langle a^{21} \rangle \cap \langle a^{10} \rangle$.

Permutation groups

4. Given the permutations $\sigma = (1 2 4), \tau = (1 3)(2 4) \in S_4$, compute the following elements:
   a. $\sigma^{-1}$  b. $\sigma \tau$  c. $\tau \sigma$  d. $\sigma^2$  e. $\sigma^2 \tau$  f. $\sigma \tau \sigma^{-1}$  g. $\tau \sigma \tau^{-1}$

5. a. Prove that a $k$-cycle in $S_n$ is an element of order $k$.
   b. Prove that when we represent a permutation as a product of disjoint cycles, the order of the permutation is the least common multiple of the lengths of these cycles.

6. Determine if $\sigma = (1 2)(1 3 4)(1 5 2), \tau = (1 2 4 3)(3 5 2 1) \in S_5$ are even or odd.

7. Prove that $A_n$ contains an $n$-cycle if and only if $n$ is odd.

I discussed these exercises with the following people: