(1) Get access to Macaulay2 in some form, possibly by downloading it onto your laptop or onto your office computer.

(2) Let \( I = \langle xz - y^2, yw - z^2, yz - xw \rangle \subseteq \mathbb{Q}[x, y, z, w] \) and let \( f = x^2y^2w^2 - y^4z^2 \). Use the division algorithm (by hand!) to determine whether or not \( f \) lies in \( I \).

(3) We use the notation \( f \% g \) to denote the remainder of \( f \) when divided by \( g \) using the division algorithm. (This is the syntax for Macaulay2 as well.) Over \( \mathbb{Q}[x, y, z] \) find an example of polynomials \( f, g_1, g_2 \) where
\[
(f \% g_1) \% g_2 \neq (f \% g_2) \% g_1.
\]
Bonus: Can you find \( f, g_1, g_2 \) such that \( (f \% g_1) \% g_2 = 0 \) but \( (f \% g_2) \% g_1 \neq 0 \)?

(4) Find an example of an ideal \( I \) where \( I \) is generated by quadrics by where \( \text{in}_>(I) \) has a minimal generator of degree at least 4. You can choose any monomial order \( > \) you like.

(5) Let \( > \) be a monomial ordering on \( S \) that respects degree, i.e. \( x^\alpha > x^\beta \) whenever \( |\alpha| > |\beta| \). For an arbitrary (i.e. not necessarily homogeneous) polynomial \( g \in S := k[x_1, \ldots, x_n] \) of degree \( d \), we define the homogenization of \( g \) by \( x_0 \) as
\[
g^h(x_0, x_1, \ldots, x_n) := x_0^d \cdot g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).
\]
For an ideal \( I \subseteq S \) define
\[
I^h := \langle g^h | g \in I \rangle \subseteq k[x_0, \ldots, x_n].
\]
Fix a monomial ordering \( > \) on \( S \), and let \( G = \{g_1, \ldots, g_r\} \) be a Gröbner basis of \( I \). Prove that
\[
I^h = \langle g_1^h, \ldots, g_r^h \rangle.
\]

(6) Using the lex term order, compute a reduced Gröbner basis (by hand!) for the ideal
\[
\langle xy - x - 2y + 2, x^2 + xy - 2x \rangle \subseteq \mathbb{C}[x, y].
\]
Use this to determine all solutions in \( \mathbb{C}^2 \) to the system of equations:
\[
\begin{align*}
xy - x - 2y + 2 &= 0 \\
x^2 + xy - 2x &= 0
\end{align*}
\]

Bonus: instead of \( \mathbb{C} \), we could have worked over a field \( k \) of positive characteristic. Over which characteristics would the Gröbner basis have been the same?