Math 234  
Chapter 14 - Extra problems

1. Let $f(x, y) = \frac{x^2y}{x^4 + y^2}$.
   
   (a) Show that $f$ has a limit as $(x, y) \to (0, 0)$ along any straight line.
   
   (b) Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

2. Let $f(x, y)$ be a twice differentiable function and consider polar coordinates $x = r \cos \theta$, $y = r \sin \theta$.
   
   (a) Show that $f_r = \cos \theta f_x + \sin \theta f_y$ and $f_\theta = -r \sin \theta f_x + r \cos \theta f_y$.
   
   (b) Show that
   $$f_{rr} = \cos^2 \theta f_{xx} + 2 \sin \theta \cos \theta f_{xy} + \sin^2 \theta f_{yy}$$
   
   and
   $$f_{\theta \theta} = r^2 \sin^2 \theta f_{xx} - 2r^2 \sin \theta \cos \theta f_{xy} + r^2 \cos^2 \theta f_{yy} - r \cos \theta f_x - r \sin \theta f_y$$
   
   (c) Conclude that
   $$f_{xx} + f_{yy} = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta \theta}$$
   
   The expression in part (c) is called the Laplacian of $f$ and is of great importance in mathematics and physics.

3. (a) Let $f$ be a differentiable function and let $\mathbf{r}(t) = x(t) \hat{i} + y(t) \hat{j}$ be a smooth curve. Show that
   $$\frac{d}{dt} f(x(t), y(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$
   
   (b) Let $f$ be a differentiable function. Show that at any point $(x_0, y_0)$ in the domain of $f$, the gradient $\nabla f(x_0, y_0)$ is perpendicular to the level curve through $f(x_0, y_0)$. 

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