§ 12.2

48. We know that
\[ |\mathbf{F}_1| = 15 \text{ and } |\mathbf{F}_2| = |\mathbf{F}_3| = 75 \]
Introducing a system of coordinates as in the picture
and using a bit of trigonometry
we see that
\[ \mathbf{F}_1 = -\cos \alpha \mathbf{i} + \sin \alpha \mathbf{j} = -\cos 75^\circ \mathbf{i} + \sin 75^\circ \mathbf{j} \]
\[ \mathbf{F}_2 = \cos \beta \mathbf{i} + \sin \beta \mathbf{j} = \cos 75^\circ \mathbf{i} + \sin 75^\circ \mathbf{j}, \text{ since } \alpha = \beta, \text{ and } \]
\[ \mathbf{F}_3 = -25 \mathbf{j} \]
Since the system is at an equilibrium, we must have
\[ \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0} \]. Therefore
\[ (-\cos 75^\circ + \cos 75^\circ) \mathbf{i} + (\sin 75^\circ + \sin 75^\circ - 25) \mathbf{j} = \mathbf{0} \]
which implies that
\[ 15 \sin \alpha = 25 \]
So, \[ \sin \alpha = \frac{1}{6} \] and hence \[ \alpha = \sin^{-1} \left( \frac{1}{6} \right) \approx 1.67 \text{ rad (or } 95.5^\circ) \]

§ 12.3

16. We introduce a system of coordinates
as in the picture.
We need to find the angle \( \theta \) between the vectors \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \).
Since \( A \) lies on the \( x \)-axis,
its coordinates are \( (x_0, 0, 0) \),
for some positive \( x_0 \).
Since \( B \) is on the \( z \)-axis,
its coordinates are \( (0, 0, z_0) \),
for some positive \( z_0 \).
But we know that the line
through \( A \) and \( B \) has a 20\% grade,
so \[ z_0 = \frac{20}{100} x_0 = \frac{1}{5} x_0 \].
Next, C will have coordinates \((0, y_0, z_1)\) for some positive \(y_0\) and \(z_1\).

Since the line through B and C has a 10% grade, we see that
\[ z_1 = z_0 + \frac{1}{10} y_0 = \frac{1}{5} x_0 + \frac{1}{10} y_0 \]

Therefore, \(A(x_0, 0, 0)\), \(B(0, 0, \frac{1}{5} x_0)\) and \(C(0, y_0, \frac{1}{5} x_0 + \frac{1}{10} y_0)\) and so \(\overrightarrow{BA} = \langle x_0, 0, -\frac{1}{5} x_0 \rangle\) and \(\overrightarrow{BC} = \langle 0, y_0, \frac{1}{10} y_0 \rangle\)

We then calculate
\[ \overrightarrow{BA} \cdot \overrightarrow{BC} = x_0 \cdot 0 + 0 \cdot y_0 + (-\frac{1}{5} x_0) \cdot \frac{1}{10} y_0 = -\frac{1}{50} x_0 y_0 \]

\[ |\overrightarrow{BA}|^2 = \sqrt{x_0^2 + \left(-\frac{1}{5} x_0\right)^2} = \sqrt{x_0^2 + \frac{1}{25} x_0^2} = \frac{\sqrt{26}}{5} x_0 \]

\[ |\overrightarrow{BC}|^2 = \sqrt{y_0^2 + \left(\frac{1}{10} y_0\right)^2} = \sqrt{y_0^2 + \frac{1}{100} y_0^2} = \frac{\sqrt{101}}{10} y_0 \]

Hence
\[ \cos \Theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} = -\frac{\frac{1}{50} x_0 y_0}{\frac{\sqrt{26}}{5} x_0 \frac{\sqrt{101}}{10} y_0} = -\frac{1}{\sqrt{26}\sqrt{101}} \]

and \(\Theta = \cos^{-1}\left(-\frac{1}{\sqrt{26}\sqrt{101}}\right) \approx 1.59\) rad (or 91.12°)