1. (7 points) Find all the local extreme values and saddle points of

\[ f(x, y) = x^3 + 3xy + y^3 \]

\[
\begin{align*}
    f_x (x, y) &= 3x^2 + 3y \\
    f_y (x, y) &= 3x + 3y^2 \\
    f_x (x, y) &= 0 \Rightarrow 3x^2 + 3y = 0 \Rightarrow x^2 + y = 0 \\
    f_y (x, y) &= 0 \Rightarrow 3x + 3y^2 = 0 \Rightarrow x + y^2 = 0 \\
    \therefore y &= -x^2 \\
    \Rightarrow x + (\text{y}^2) &= 0 \Rightarrow x^3 + x = 0 \Rightarrow x(x^2 + 1) = 0 \\
    \therefore y &= -x^2 \\
    \therefore x = 0 \text{ or } x^3 &= -1 \Leftrightarrow -1 \\
\end{align*}
\]

So critical points: \((x, y) = (0, 0), (-1, -1)\)

\[
\begin{align*}
    f_{xx} (x, y) &= 6x \\
    f_{xy} (x, y) &= 3 \\
    f_{yy} &= 6y \\
    D(x, y) &= f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 = 36x - y^3 \\
\end{align*}
\]

\[
\begin{align*}
    D(0, 0) &= -9 < 0 \Rightarrow \text{saddle point at } (0, 0) \\
    D(-1, -1) &= 36 - 9 = 27 > 0 \Rightarrow \text{local maximum of } f(-1, -1) \\
    f_{xx} (-1, -1) &= -6 < 0 \text{ at } (-1, -1)
\end{align*}
\]
2. (4 points) Let $x, y$ and $z$ be three numbers whose sum is equal to 6. What is the smallest value the sum of their squares can have? Why?

We need to minimize $f(x, y, z) = x^2 + y^2 + z^2$
subject to the constraint $x + y + z = 6$
so let $g(x, y, z) = x + y + z$

$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$
$\nabla g(x, y, z) = \langle 1, 1, 1 \rangle$

$2x = 2 \Rightarrow x = 1$
$2y = 2 \Rightarrow y = 1$
$2z = 2 \Rightarrow z = 0$
$x + y + z = 6 \Rightarrow 1 + 1 + 0 = 2$

The method of Lagrange multiplies guarantees there is a maximum or a minimum when $(x, y, z) = (2, 2, 2)$.
To see that this is a minimum and not a maximum, choose another triple satisfying $x + y + z = 6$, say $(x, y, z) = (1, 1, 4)$ and compute the sum of the squares: $1^2 + 1^2 + 4^2 = 18 > 12 = 2^2 + 2^2 + 2^2$
So the smallest value of the sum of the squares is 12 (when $x = y = z = 2$)