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Note: Show all your work. No calculators allowed.

1. (10 points) Let \( f(x, y) = x^3 - 3xy + y^3 \).

   (a) Find the critical points of \( f \).
   (b) Use the quadratic approximation to classify the critical points you found as local minima, local maxima, or saddle points.

Solution

(a) We first compute the partial derivatives

\[
\begin{align*}
    f_x(x, y) &= 3x^2 - 3y \\
    f_y(x, y) &= -3x + 3y^2
\end{align*}
\]

and then solve for critical points:

\[
\begin{align*}
    3x^2 - 3y &= 0 \\
-3x + 3y^2 &= 0 \\
\end{align*} \iff \begin{align*}
    x^2 - y &= 0 \\
-x + y^2 &= 0 \\
\end{align*} \iff \begin{align*}
    y &= x^2 \\
    x &= y^2
\end{align*} \iff \begin{align*}
    y &= x^2 \\
    x &= x^4
\end{align*} \iff \begin{align*}
    y &= x^2 \\
    x(1 - x^3) &= 0 \\
\end{align*} \iff x = 0, x = 1
\]

When \( x = 0 \), the first equation gives \( y = 0 \), while when \( x = 1 \), we get \( y = 1 \). Hence the critical points are \((0,0)\) and \((1,1)\).
(b) The quadratic approximation at a critical point \((x_0, y_0)\) gives

\[
f(x, y) - f(x_0, y_0) \\ \approx \frac{1}{2} f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2} f_{yy}(x_0, y_0)(y - y_0)^2.
\]

Since \(f_{xx}(x, y) = 6x\), \(f_{xy}(x, y) = -3\) and \(f_{yy}(x, y) = -6y\), we have

\[
f(x, y) - f(0, 0) \approx \frac{1}{2} f_{xx}(0, 0) x^2 + f_{xy}(0, 0) xy + \frac{1}{2} f_{yy}(0, 0) y^2
\]
\[= -3xy
\]

But the quadratic form in the last display is clearly indefinite, so \(f\) has a saddle point at \((0, 0)\).

At \((1, 1)\) we have

\[
f(x, y) - f(1, 1)
\]
\[\approx \frac{1}{2} f_{xx}(1, 1)(x - 1)^2 + f_{xy}(1, 1)(x - 1)(y - 1) + \frac{1}{2} f_{yy}(1, 1)(y - 1)^2
\]
\[= \frac{1}{2} 6u^2 - 3uv + \frac{1}{2} 6v^2, \quad \text{where} \quad u = x - 1, \ v = y - 1
\]
\[= 3(u^2 - uv + v^2)
\]
\[= 3 \left[ u^2 - uv + \frac{v^2}{4} - \frac{v^2}{4} - v^2 \right]
\]
\[= 3 \left[ (u - \frac{v}{2})^2 + \frac{3v^2}{4} \right].
\]

This quadratic form is positive definite, so \(f\) has a local minimum at \((1, 1)\).